Fundamental matter, meson spectroscopy and non-critical string/gauge duality

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ABSTRACT: We discuss the incorporation of quarks in the fundamental representation of the color group into the non-critical string/gauge duality. We focus on confining theories and address this question using two different approaches: (i) by introducing flavor probe branes and (ii) by deriving backreacted flavored near extremal gravity backgrounds. In the former approach we analyze the near extremal AdS_6 model with D4 and anti-D4 probe flavor branes included. We study the meson spectrum and discuss the role played by the constituent quark mass, related to the integration constant that defines the embedding. As for the second approach we derive a class of flavored $AdS_{n+1} \times S^k$ black hole solutions. In particular we write down the flavored AdS_6 and AdS_5 black holes and the near extremal $AdS_5 \times S^1$ backgrounds. We analyze several gauge dynamical properties associated with these models.

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1. Introduction

The program of constructing the string theory holographic dual to QCD has made a significant progress in recent years. However, there are still several major open questions that have to be addressed. These include in particular the following two challenges:

- A full understanding of the consistent incorporation of flavored quarks in the fundamental representation of the color group;

- Cleaning the spectrum from the undesired KK modes associated with the compact transverse directions.

In this paper we address these two issues in the context of non-critical (spacetime dimension less than ten) string/gauge dualities [1]–[5].

The identification of the particles of confining gauge theories in their dual supergravity (string) models can be achieved using the following three different approaches:

1. For gravity backgrounds that incorporate bulk fields which correspond to the assigned states of the boundary gauge theory, one computes the spectrum of the fluctuations of those bulk fields and deduces from it the spectrum of the dual particles.

2. For states of the gauge theory that do not have associated bulk fields in the dual gravity theory, one introduces probe branes in such a way that their fluctuations play the role of the desired states.

3. For states of spin higher than two one has to go beyond the gravity limit and consider string configurations that describe the particles.

A prototype of the identification of the first kind is the four-dimensional glueball spectrum extracted from the fluctuations of the dilaton (and other fields) in confining backgrounds [6], such as the model based on near extremal D4 branes.

To dualize the fundamental quarks one can either derive a string theory or its gravity low-energy effective action that incorporates fields dual to the quarks or else use approach 2 of introducing flavor probe branes. Since most of the known supergravity backgrounds do not incorporate fields dual to fundamental quarks, it was the idea of [7] to introduce probe branes in such a way that strings stretching between them and the original branes have the properties of flavored fundamental quarks. In the original model [7] N_f D7 probe branes were incorporated in the $AdS_5 \times S^5$ supergravity and the strings between the D3 and D7 branes were the duals of hyper-multiplets of the resulting N=2 SQCD theory in the fundamental representation of both the color and flavor groups. The fluctuations of the probe branes were shown [8] to correspond to the mesons of this theory which is in a Coulomb phase. The first attempt to describe mesons of a confining gauge theory was made in [9] where D7 probe branes were introduced into the Klebanov-Strassler model [10], and the spectrum of massive vector and pseudoscalar mesons was computed. A much simpler and more illuminating model was proposed in [11] which is based on the addition of D6 flavor branes to the model of near extremal D4 branes [12]. For further work devoted to the addition of flavors in the context of critical string duality, see, for instance, [13]–[15]. Related to this context, several holographic models for hadrons have also been built, where an *ad hoc* IR cutoff is introduced in AdS_5 in order to make the dual theory confining and with a mass gap [16] (see also [17]), achieving notable success for their quantitative predictions. However, it is of obvious interest to obtain this kind of five-dimensional actions from string theory.

Recently, progress in this direction was made by incorporating D8,$\bar{\text{D8}}$ branes in the near extremal D4 brane background [12], resulting in a model with a non-abelian flavor

symmetry and flavor chiral symmetry breaking [18]. In particular the corresponding Goldstone bosons were identified. Furthermore, ratios of the masses of certain vector and scalar mesons were computed in this model and were compared to the experimental values yielding a fairly nice agreement. In a later work, the same authors computed in their model decay rates of certain mesons [19]. All these models are examples of the second type of approach mentioned above.

Glueballs and mesons of spin higher than two cannot be described by a low energy gravitational theory simply because the latter does not include fields of spin higher than two. The hadronic spectrum of confining gauge theories however does include hadrons of higher spin and the lightest ones for each spin furnish a Regge trajectory, namely, they obey the relation $J = \alpha' M^2 + \alpha_0$. It is well known that this kind of relation between energy and angular momentum is the characteristic behavior of open strings in flat spacetime. Recently, it has been shown that string configurations that reside in the vicinity of the "wall" of confining backgrounds also admit the Regge trajectory behavior. In particular in [20] using a semi-classical quantization this was shown for closed strings, the duals of glueballs, in the context of the Klebanov-Strassler [10] and Maldacena-Núñez [21] models, and in [22] for the near extremal D4 branes background. Regarding high spin mesons, it was shown in [23] that the open strings that describe them are equivalent to open strings with massive endpoints spinning in flat space.

These developments address the first challenge mentioned above. As for the second, the most naive approach to the problem of the KK modes is the use of non-critical string theories where the KK (or at least most of them) are absent a priori. The idea to employ a non-critical string theory as the string theory of QCD was raised in [1]. This proposal was further investigated in several other papers [24]. Recently, a few families of non-critical backgrounds were identified and analyzed [2], including in particular a class of models with spacetimes of the form $AdS_{n+1} \times S^k$, a constant dilaton and an F_{n+1} RR form. "Measurements" of various properties were performed in the AdS_6 black hole "laboratory" in [4].

Non-critical supergravities suffer from the fact that they have necessarily finite curvature in units of the string scale due to the non-critical term in their action. However, there is some evidence that the basic structure of spacetime is not modified due to higher order curvature corrections in a similar manner to what occurs for strings on group manifolds. In this paper we follow this approach and consider non-critical gravity models. On the other hand, recent developments using a worldsheet approach to non-critical strings can be found in [25].

To introduce fundamental quarks into non-critical models, one can again follow one of the two alternatives above: to derive a fully backreacted gravity background or to analyze a system with probe branes. In the present work we discuss both options.

The "laboratory" to which we add flavor probe branes is the AdS_6 non-critical black hole solution [2]. This model was shown [4] to reproduce some properties of 4d non-supersymmetric YM theory like an area law for the Wilson loop, a mass gap in the glueball spectrum etc. In this work we introduce flavor in this setup by adding D4,$\bar{\text{D}}$4 probe branes. This is similar to the D8,$\bar{\text{D}}$8 model of [18] in critical dimension and it inherits from

it many qualitative features. We derive the profile of the probe branes deduced from an action with or without a CS term. We analyze the pseudoscalar and vector fluctuations on the worldvolume of the probe branes and extract the spectrum of the corresponding pseudoscalar and vector mesons. Similarly to [18] we find massless Goldstone bosons. We generalize the solutions of [18] by allowing the lowest point of the probe brane to be away from the "wall" of the spacetime. We assign to the difference between the former and the latter a "mass parameter" m_q. We find that the masses of the low spin mesons increase monotonically with this mass parameter (for a probe action that includes a CS term this increase takes place only after a certain minimal value of m_q). We compute the ratios of the masses of various mesons and find that the agreement with the measured values is of the same order as those of the critical model [18]. It turns out, however, that even for a non-trivial mass parameter the "pions" remain massless. This result indicates that, in the terminology of QCD, m_q does not correspond to the current algebra mass of the quarks (see section 3.5 for a discussion). We then analyze the spectrum of classical spinning open string configurations and find that as for the critical case, one can describe the mesons as spinning open strings in flat spacetime with massive endpoints whose mass is m_q.

As for the fully backreacted effective gravitational actions, some solutions have been computed in critical string theory [26]. In several non-critical models [3, 27, 5], open string degrees of freedom that are dual to quarks in the color and flavor fundamental representation have been incorporated by adding a DBI term to the gravity action. The Klebanov-Maldacena (KM) [3] model is an $AdS_5 \times S^1$ supergravity background which is conjectured to be the holographic dual of the IR fixed point of the $N = 1$ SQCD in the so called "conformal window". In [5] two other flavored models were derived in 5d and 8d corresponding to a QCD-like theory with $N = 0$ and $N = 1$ supersymmetries respectively.

A natural question to address at this stage is the derivation of a background which is dual to a flavored gauge theory which is not conformal but rather confining, for instance non-supersymmetric QCD. One can address this challenge by either starting with unflavored non-conformal backgrounds and add quarks to them, or by taking a conformal flavored model, break conformal invariance and turn on confinement. In this paper we do both. We derive non-critical non-conformal flavored backgrounds by starting with the non-conformal near extremal AdS_6 (AdS_5) model, which describes non-critical near extremal D4 (D3) branes. We then introduce N_f flavor branes and write down the system of equations that describes the fully backreacted solution of the gravity equations of motion. In the AdS_6 case, with D4 flavor branes smeared over the entire "thermal circle" the only solution we could determine explicitly is eventually a conformal solution, the T-dual of the KM model. On the other hand for space-filling D5 branes we do find a flavored AdS_6 black hole solution. As in the original proposal of [12] upon turning on large temperature or reducing the radius of the thermal circle we end up with a four-dimensional low-energy effective action. Unfortunately, this theory includes only the pure YM theory since the flavored fundamental quarks also acquire mass of the order of the temperature and decouple from the low-energy effective theory. We then generalize the KM model and derive a family of flavored conformal supergravity solutions in D dimensions of the form

$AdS_{n+1} \times S^k$, by incorporating N_f space-filling DBI terms into the gravity action. We also derive the corresponding black hole solutions, and in particular the near extremal limit of the KM model $AdS_5 \times S^1$ [3].

We address the issue of the "phenomenology" of the flavored near extremal models that are dual to four-dimensional gauge systems. We present several arguments about the identification of the various phases of the latter and then extract gauge dynamical properties such as the Wilson loop, the spectrum of glueballs and the spectrum of mesons.

The paper is organized as follows: after this introduction, in section 2 the general setup of the near extremal AdS_6 non-critical background is described. We then discuss qualitatively the options to introduce flavor branes into this model. Section 3 is devoted to a quantitative analysis of the system when D4 probe branes are incorporated. The solution for the classical probe profile is written down and the spectrum of the fluctuations of the gauge fields as well as the one of (pseudo) scalar fields is extracted. This includes the identification of the Goldstone bosons, the computation of the masses of the low-lying mesons and a discussion of the dependence of the these features on the CS coefficient and on m_q, the mass parameter of the probe profile. We then analyze the configurations of open spinning strings. It is shown that similarly to the result reported in [23], these configurations behave like open strings in flat spacetime with massive endpoints. In section 4 we attempt at finding a backreacted solution of N_f flavor D4 branes that are smeared along the thermal cycle. The only constant dilaton solution the system admits is an $AdS_5 \times S^1$ geometry which is T-dual to the KM solution. We devote section 5 to the study of solutions associated with space-filling probe branes in any number D of dimensions and with an S^k transverse space. Both conformal solutions and their black hole generalizations are derived. In sections 5.1–5.3 we present separately the most interesting cases. In section 6 the phenomenology of the gauge theories dual to some of these backgrounds is extracted. We end up with a summary and a list of several open questions in section 7.

2. Towards a non-critical string dual of QCD

The six-dimensional background we consider in this section is the non-critical version [2] of the ten-dimensional black hole background dual to thermal/pure Yang-Mills in four dimensions [12]. In both the critical and non-critical cases, the gravity background is generated by near-extremal D4-branes wrapped over a circle with anti-periodic boundary conditions. In the non-critical case it takes the form of a static black hole embedded inside six-dimensional anti-de Sitter space. The anti-periodic boundary conditions project massless fermions out of the spectrum, and the only surviving fermionic degrees of freedom are excited KK modes. In turn, this also gives a mass to all the scalars via one-loop corrections. At high energies, therefore, the D4-brane dynamics describes a five-dimensional gauge theory at the same finite temperature as the black hole temperature, whereas at low-energies, the KK modes cannot be excited and the theory is effectively pure Yang-Mills in four dimensions [12].

The background geometry consists of a constant dilaton ϕ, a RR six-form field strength $F_{(6)}$ and a metric [2]:

$$ds_6^2 = \left(\frac{u}{R_{AdS}}\right)^2 dx_{1,3}^2 + \left(\frac{R_{AdS}}{u}\right)^2 \frac{du^2}{f(u)} + \left(\frac{u}{R_{AdS}}\right)^2 f(u)d\eta^2 \tag{2.1}$$

$$F_{(6)} = Q_c \left(\frac{u}{R_{AdS}}\right)^4 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge du \wedge d\eta \tag{2.2}$$

$$e^\phi = \frac{2\sqrt{2}}{\sqrt{3}Q_c} \ , \qquad R_{AdS}^2 = \frac{15}{2} \tag{2.3}$$

with:

$$f(u) = 1 - \left(\frac{u_\Lambda}{u}\right)^5 \tag{2.4}$$

where Q_c is proportional to the number of color D4 branes and, to avoid a conical singularity at the origin, the coordinate η needs to be periodic:

$$\eta \sim \eta + \delta\eta \ , \qquad \delta\eta = \frac{4\pi R_{AdS}^2}{5u_\Lambda} \ . \tag{2.5}$$

We also define M_Λ as the typical mass scale below which the theory is effectively four-dimensional:

$$M_\Lambda = \frac{2\pi}{\delta\eta} = \frac{5}{2}\frac{u_\Lambda}{R_{AdS}^2} \ . \tag{2.6}$$

Several interesting questions about the gauge theory dual to this background can be addressed by analyzing the closed string spectrum of non-critical string theory on (2.1) [4]. Despite the fact that one expects large stringy corrections to affect every calculation on the non-critical gravity side, the results obtained in [4] are at least the same order of magnitude as those given by experiments or lattice calculations, showing therefore that also in this case the approximation to the complete string theory being considered is not meaningless. This encouraged us to push the study of this background a little further, and introduce fundamental degrees of freedom in the gauge theory.

Since the gauge theory is non-supersymmetric, there seem to be two different ways to do this, that is by adding additional D4 or D5-branes, which we will generically call flavor probe branes. In fact, while both ways of proceeding provide fundamental degrees of freedom, the interpretation of the resulting gauge theory turns out to be quite different in the two cases. It seems natural to argue that the low-energy limit of the gauge theory contains massless fundamentals only in the case where these were added through D4-branes. The reason is straightforward, and relies on the same argument that gives non-supersymmetric, four-dimensional, pure Yang-Mills as the low-energy limit of the five-dimensional theory living on the wrapped D4 color branes [12, 2]. As mentioned above, to project out the massless gaugino from the low-energy spectrum, we have to impose anti-periodic boundary conditions over the thermal cycle [12]. But when we add flavor D5-branes (which obviously need to wrap the S^1), the same mechanism gives mass to the quarks of the four-dimensional theory, leaving us with the massless degrees of freedom of a four-dimensional, non-supersymmetric pure Yang Mills theory plus massive quarks and

gauginos. As a confirmation, we will see in section 5.1 that indeed when a large number of D5-branes is added to the color D4-branes, the new background that is obtained has the same form as the AdS_6 black hole (2.1), but with different coefficients, signaling that new massive UV states are being integrated out when going to the IR. Nonetheless, the addition of D5-branes is still interesting because it allows us to explore QCD at finite temperature.

For the addition of D4-branes, instead, the enforcing of anti-periodic boundary conditions over the circle should not affect the fundamental degrees of freedom which are introduced by the addition of the new flavor branes. This is certainly the case in critical string theory, where the addition of flavor branes transverse to the S^1 circle provides massless quarks in the low-energy theory [8, 18]. We argue that the same is also true for the present non-critical string theory. Therefore we expect that the introduction of flavor D4-branes provides an alternative scenario for the study of a string dual to QCD.

In both the D4 and D5 cases the flavor branes need to fill the whole four-dimensional Minkowski space and stretch along the radius u up to infinity. This last condition makes the gauge coupling on the flavor branes very small, freezing its gauge group to a rigid symmetry group, that is the QCD $U(N_f) \times U(N_f)$ flavor group.

One relevant feature of the analysis of QCD in terms of a string dual is the ability to reproduce the spontaneous chiral symmetry breaking of the $U(N_f) \times U(N_f)$ flavor group into the diagonal $U(N_f)$ when all the quarks are massless.

The two factors of the flavor group are obtained by introducing two different stacks of flavor branes, that is one of branes and the other one of anti-branes, in such a way that strings hanging between a color D4-brane and a flavor brane transform as quarks, while strings hanging between a color D4 and a flavor anti-brane transform as anti-quarks. The overall system of flavor branes is uncharged, as in [3, 18, 5]. The chiral symmetry breaking is achieved by a reconnection of the brane-anti-brane pairs [18].

3. Probe D4 flavor branes on the AdS_6 black hole

As argued above, the most natural way of adding (few) flavors to the AdS_6 black hole background [2] is to include D4-probe branes extended along the Minkowski directions and stretching to infinity in the radial one. The picture is very similar to the approach to QCD in [18] where flavors are added to the ten-dimensional background of [12] by considering D8, $\bar{D}8$ flavor branes. The main difference is the absence of the transverse S^4 of the critical case, which plays no physical role.

We consider the action:

$$S_{D4} = -T_4 \int d^5x \, e^{-\phi} \sqrt{-\det(\hat{g} + 2\pi\alpha' F)} + T_4 \, \tilde{a} \int \mathcal{P}(C_{(5)}) \qquad (3.1)$$

where $\hat{g}$ stands for the pullback of the metric on the probe worldvolume. It has to be pointed out that this action does not account for all the degrees of freedom of the system. In particular, there are strings connecting the flavor D4 and $\bar{D}4$ branes, but we lack a reasonable description of an effective action for them. Nevertheless, we do not expect them to destabilize the setup. We leave the WZ coupling in the action as a direct generalization

of the well known one in the critical setup, although, as far as we know, it is not well understood what should be written in this two-derivative approximation to the non-critical setup. One should take $\tilde{a} = 0$ if this WZ coupling is not present at all whereas $\tilde{a} = 1$ would be the direct naive generalization from the ten-dimensional theory.

We consider x_i, η as the worldvolume coordinates of the probe brane and allow for $u = u(\eta)$. We find the lagrangian density (prime denotes derivative with respect to η):

$$\mathcal{L} = -T_4 e^{-\phi} \left[\left(\frac{u}{R_{AdS}} \right)^4 \sqrt{ \left(\frac{u}{R_{AdS}} \right)^2 f(u) + \left(\frac{R_{AdS}}{u} \right)^2 f^{-1}(u) u'^2 } - a \left(\frac{u}{R_{AdS}} \right)^5 \right] \quad (3.2)$$

where we have defined a new constant $a = \frac{2}{\sqrt{5}}\tilde{a}$ in order to simplify the notation. Since the lagrangian does not explicitly depend on η, the quantity $u'\frac{\partial \mathcal{L}}{\partial u'} - \mathcal{L}$ is a constant, and we can write:

$$\left(\frac{u}{R_{AdS}} \right)^5 \left[\frac{f(u)}{\sqrt{ f(u) + \left(\frac{R_{AdS}}{u} \right)^4 \frac{u'^2}{f(u)} }} - a \right] = \left(\frac{u_0}{R_{AdS}} \right)^5 \left[f^{\frac{1}{2}}(u_0) - a \right] \quad (3.3)$$

where we have defined u_0 as the minimal value of the radial coordinate reached by the probe brane. Therefore, the classical static embedding[1] is given by:

$$\eta_{st}(u) = \int_{u_0}^{u} \frac{(u_0^5 f^{\frac{1}{2}}(u_0) - au_0^5 + au^5)\ du}{\left(\frac{u}{R_{AdS}} \right)^2 f(u) \sqrt{u^{10} f(u) - (u_0^5 f^{\frac{1}{2}}(u_0) - au_0^5 + au^5)^2}} \quad (3.4)$$

Notice that tuning the constant of integration u_0 modifies the boundary conditions at infinity, namely $\eta(\infty)$. On general grounds, we may expect that different u_0 account for different dual theories and we interpret this constant as being related to the constituent mass of the quarks, as will be discussed below in section 3.5. In particular, for $a = 0$, $\eta(\infty)|_{u_0=u_\Lambda} = \delta\eta/4$ (where $\delta\eta$ is as defined in (2.5)) and $\eta(\infty)|_{u_0 \to \infty} \to 0$, as in [18]. In the probe approximation which we use here, $u_0 \neq u_\Lambda$ is a stable solution of the equation of motion and as will be shown shortly the fluctuations around it are non-tachyonic. However, it might be that because the D4 flavor branes are codimension one, by going beyond the probe approximation the backreaction with the F_6 field strength associated with the flavor branes may become important and a matter of concern.[2]

In the following, we consider small fluctuations around this embedding which correspond to the physical particles composed by the fundamentals. The analysis is similar to that in [18] but here we keep the constant of integration u_0 generic. In section 3.5 we will discuss its relation to the mass of the quarks. Indeed, we can define a mass parameter m_q as the energy of a string connecting the color and flavor branes [28] (stretching from the horizon to u_0), i.e.

$$m_q = E_s = \frac{1}{2\pi\alpha'} \int_{u_\Lambda}^{u_0} \sqrt{-g_{00} g_{uu}}\, du = \frac{1}{2\pi\alpha'} \int_{u_\Lambda}^{u_0} f^{-\frac{1}{2}}(u)\, du. \quad (3.5)$$

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[1]$a < 1$ is required for consistency.

[2]We thank Igor Klebanov for pointing this out to us.

3.1 Fluctuations of the gauge field

The equations of motion for the gauge fields living on the D4 flavor brane, that follow from the probe action (3.1), are:

$$\partial_m \left(\sqrt{-\det \hat{g}} \, F^{mn} \right) = 0 \,. \tag{3.6}$$

By splitting the field strength into its four-dimensional and internal components, we can write:

$$u^4 \gamma^{\frac{1}{2}} \partial_\mu F^{\mu\nu} - \partial_u (u^4 \gamma^{\frac{1}{2}} F^{\nu u}) = 0 \tag{3.7}$$

$$\partial_\mu F^{\mu u} = 0 \,, \tag{3.8}$$

where we have defined

$$\gamma = \frac{u^8}{f(u) u^{10} - (u_0^5 f^{\frac{1}{2}}(u_0) - a u_0^5 + a u^5)^2} \,. \tag{3.9}$$

At the same time, we can expand the gauge field in the following way

$$A_\mu(x^\mu, u) = \sum_n B_\mu^{(n)}(x^\mu) \psi_{(n)}(u)$$

$$A_u(x^\mu, u) = \sum_n \varphi^{(n)}(x^\mu) \phi_{(n)}(u) \tag{3.10}$$

and for convenience we also define

$$F_{\mu\nu}^{(n)}(x^\mu) = \partial_\mu B_\nu^{(n)} - \partial_\nu B_\mu^{(n)} \tag{3.11}$$

Under the decomposition (3.10), equation (3.8) reads:

$$\sum_n \left(\tilde{m}_n^2 \varphi^{(n)}(x^\mu) \phi_{(n)}(u) - (\partial_\mu B_{(n)}^\mu (x^\mu)) \partial_u \psi_{(n)}(u) \right) = 0 \tag{3.12}$$

where $\tilde{m}_n$ is the four-dimensional mass of the field $\varphi^{(n)}$: $\eta^{\mu\nu} \partial_\mu \partial_\nu \varphi^{(n)} = \tilde{m}_n^2 \varphi^{(n)}$. We see immediately that choosing the Lorentz gauge $\partial_\mu B_{(n)}^\mu = 0$ for the four-dimensional gauge fields sets to zero all the massive scalars $\varphi^{(n)}$. On the contrary, the massless field $\varphi^{(0)}$ survives, regardless of the value of the free parameter u_0 appearing in γ.

Let us consider now the other equation of motion (3.7). Substituting (3.10) in, it is easy to show that (3.7) fixes the form of $\phi_{(0)}$ to be

$$\phi_0 \sim \frac{\gamma^{\frac{1}{2}}}{u^2} \,, \tag{3.13}$$

and also imposes the defining equation for the modes $\psi_{(n)}(u)$:

$$-\gamma^{-\frac{1}{2}} \partial_u (u^2 \gamma^{-\frac{1}{2}} \partial_u \psi_{(n)}) = R_{AdS}^4 m_n^2 \psi_{(n)} \tag{3.14}$$

where m_n is the mass of the four-dimensional vector field $B_\mu^{(n)}$: $\eta^{\nu\rho} \partial_\nu \partial_\rho B_\mu^{(n)} = m_n^2 B_\mu^{(n)}$. The last ingredient we still miss to make sense of the expansions (3.10) are orthonormality

conditions for the modes $\psi_{(n)}$ and $\phi_{(0)}$. The natural choice is to require that the kinetic terms in the four-dimensional effective action are canonically normalized, and therefore the conditions we are after, read

$$C\,(2\pi\alpha')^2 \int du\gamma^{\frac{1}{2}}\psi_{(n)}\psi_{(m)} = \delta_{nm}\ , \tag{3.15}$$

$$C\,(2\pi\alpha')^2 \int du\frac{u^2}{R_{AdS}^4}\gamma^{-\frac{1}{2}}\phi_{(0)}\phi_{(0)} = 1\ , \tag{3.16}$$

where:

$$C = T_4 e^{-\phi}R_{AdS}\,. \tag{3.17}$$

It is straightforward to show that the orthogonality condition (3.15) is consistent with equation (3.14), and that the mode ϕ_0 in (3.13) is normalizable.

Near infinity, the set of orthonormal functions $\psi_{(n)}$ solving equation (3.14) behaves as:

$$\lim_{u\to\infty}\psi_{(n)} \sim C_1 + \frac{C_2}{u^2} \tag{3.18}$$

Normalizability (3.15) fixes $C_1 = 0$. Moreover, the worldvolume coordinate u describes separately the two halves of the brane and, in order to be smooth at the origin, one has to require that ψ is an odd or even function on the worldvolume of the probe. These two conditions can be simultaneously satisfied only for certain values of m_n, generating, then, a discrete spectrum. We leave its analysis for section 3.3.

Finally, the effective four-dimensional flat space action for the fluctuations of the gauge field A^m living on the probe D4-brane can be written as:

$$S = -\int d^4x \left[\frac{1}{2}\partial_\mu\varphi^{(0)}\partial^\mu\varphi^{(0)} + \sum_{n\geq 1}\left(\frac{1}{4}F_{\mu\nu}^{(n)}F^{\mu\nu\,(n)} + \frac{1}{2}m_n^2 B_\mu^{(n)}B^{\mu\,(n)}\right)\right]\,. \tag{3.19}$$

The field $\phi_{(0)}$ can be associated to $\partial_u\psi_{(0)}$ where $\psi_{(0)}$ is the zero mode of equation (3.14); the associated four-dimensional field $\varphi^{(0)}$ corresponds to the Goldstone boson. As in [18], one can also go to the $A_u = 0$ gauge where there are no $\phi_{(n)}$ and the Goldstone boson gets encoded in the boundary ($u \to \infty$) value of A_μ. Notice that this massless mode exists for any value of u_0. We will comment on this in section 3.5.

3.2 Fluctuations of the embedding

We now consider small fluctuations around the static embedding (3.4), which are scalars in the four-dimensional theory:

$$\eta = \eta_{st}(u) + \xi(x^\mu, u) \tag{3.20}$$

Keeping up to quadratic terms in ξ and discarding a total derivative, the expansion of (3.1) yields:

$$S_{D4} = -\frac{1}{2}C \int d^4x du \left[\gamma^{-\frac{3}{2}}\frac{u^4}{R_{AdS}^8}(\partial_u\xi)^2 + \gamma^{-\frac{1}{2}}\frac{u^2}{R_{AdS}^4}\eta^{\mu\nu}\partial_\mu\xi\partial_\nu\xi\right] \tag{3.21}$$

The equation of motion for the scalar ξ reads then:

$$R_{AdS}^4 \gamma^{-\frac{1}{2}} u^2 \eta^{\mu\nu} \partial_\mu \partial_\nu \xi + \partial_u(\gamma^{-\frac{3}{2}} u^4 \partial_u \xi) = 0 \tag{3.22}$$

As for the vector field above, we make an expansion of the fluctuation field:

$$\xi = \sum_{n=1}^{\infty} \mathcal{U}^{(n)}(x^\mu) \sigma_{(n)}(u) \tag{3.23}$$

where the functions $\sigma_{(n)}(u)$ are defined by the second order equation

$$-u^{-2} \gamma^{\frac{1}{2}} \partial_u(u^4 \gamma^{-\frac{3}{2}} \partial_u \sigma_{(n)}) = R_{AdS}^4 m_n'^2 \sigma_{(n)} \tag{3.24}$$

which follows directly from the equation of motion (3.22). Here m_n' are the possible mass eigenvalues for the four-dimensional scalars $\mathcal{U}^{(n)}$: $\eta^{\mu\nu} \partial_\mu \partial_\nu \mathcal{U}^{(n)} = m_n'^2 \mathcal{U}^{(n)}$.

By substituting the expansion (3.23) into the action (3.21), one finds the effective action for a tower of four-dimensional massive scalar mesons:

$$S = \frac{1}{2} \int d^4x \sum_n \left[(\partial_\mu \mathcal{U}^{(n)})^2 + m_n'^2 (\mathcal{U}^{(n)})^2 \right] \tag{3.25}$$

where the normalization of the fields $\sigma_{(n)}$ has been chosen in order for the kinetic term to be canonically normalized:

$$C \int \gamma^{-\frac{1}{2}} \frac{u^2}{R_{AdS}^4} \sigma_{(m)} \sigma_{(n)} = \delta_{mn} \tag{3.26}$$

It is now easy to see that the asymptotic behavior of $\sigma_{(n)}$ is of the form:

$$\lim_{u \to \infty} \sigma_{(n)} \sim C_1 + \frac{C_2}{u^6} , \tag{3.27}$$

so normalizability (3.26) requires $C_1 = 0$ and, again, the mass spectrum m_n' is discrete.

3.3 Numerical analysis and discussion

In this section we proceed to a numerical study of equations (3.14) and (3.24).

Starting with the vectors (3.14), notice that two kinds of boundary conditions can be imposed on the functions $\psi_{(n)}(u)$ at the bending point of the probe brane $u = u_0$, which lead to different parity and charge conjugation for the corresponding mesons (we use J^{CP} notation):

$$\partial_u \psi_{(n)}(u)\big|_{u=u_0} = 0 \quad \to \quad 1^{--} \quad (\text{odd } n) \tag{3.28}$$
$$\psi_{(n)}(u_0) = 0 \quad \to \quad 1^{++} \quad (\text{even } n) \tag{3.29}$$

The arguments to deduce P and C are completely analogous to those in [18] and will not be repeated here. Notice, however, that we are assuming a WZ coupling on the worldvolume of the flavor brane to the Hodge dual of the RR-form, i.e. $\int_{D4} {}^*F_{(6)} \omega_5(A)$ where $d\omega_5 = F^3$.
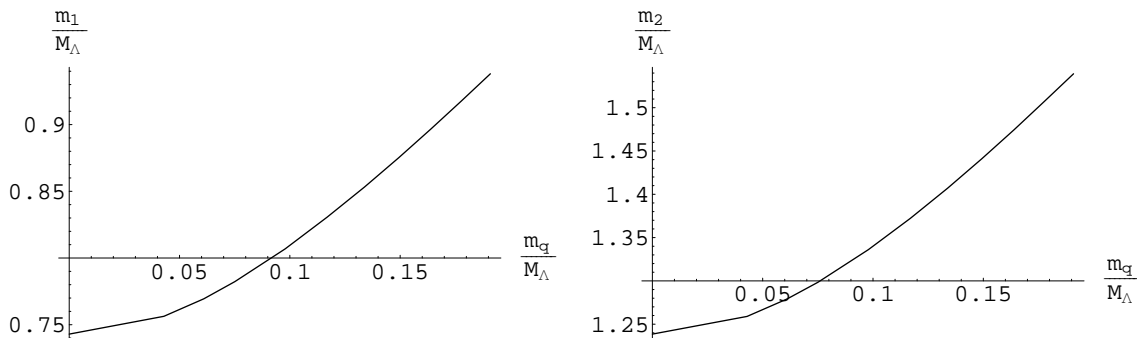
Figure 1: Plots of the masses of the two lightest vector mesons varying the constant u_0. $a = 0$ has been taken.
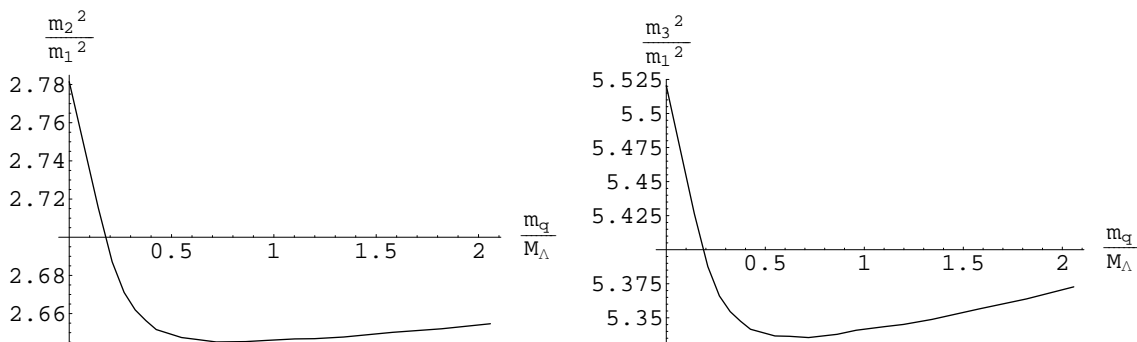

Figure 2: Quotients of the squared masses of the vector mesons for different values of the bare mass of the quarks (we have set $a = 0$). The experimental values of such quotients for the corresponding QCD mesons are: $\frac{m_2^2}{m_1^2} = 2.51$, $\frac{m_3^2}{m_1^2} = 3.56$.
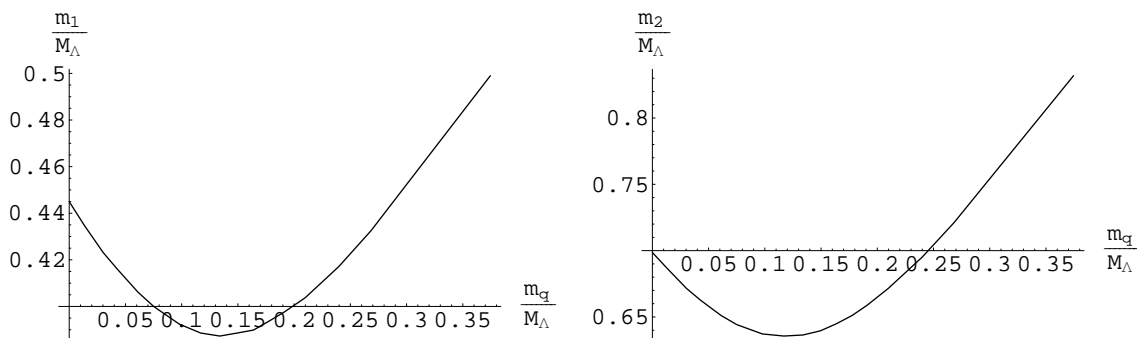

Figure 3: Plots of the masses of the two lightest vector mesons computed with $\tilde{a} = 1$.

Let us start by setting $a = 0$. Then, the results are summarized in figures 1 and 2. In particular, notice that the meson masses grow monotonically with the quark mass parameter. On the contrary, see figure 3, if we set $a \neq 0$, we find the feature that the meson masses decrease when m_q grows (at least for small m_q).

The study of scalar and pseudoscalar mesons using equation (3.24) is a bit more subtle. The choice of u as a worldvolume coordinate and η as the excited scalar is ill-defined near
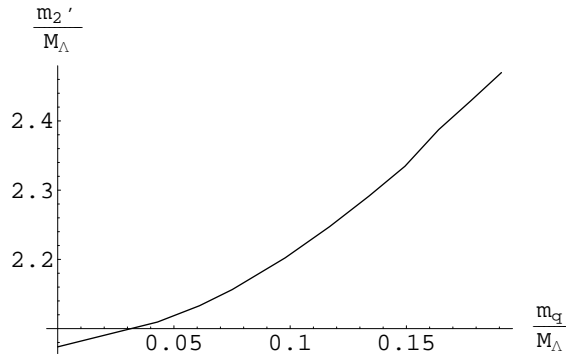
Figure 4: The mass of the next to lightest scalar meson vs. the quark mass ($a = 0$).

the tip of the probe brane, where the boundary conditions have to be imposed. In fact, this formalism does only allow us to find half of the modes: those in which the point at $u = u_0$ is fixed, i.e. the odd excitations on the brane, which correspond to 0^{--} mesons, yielding the $m_n'^2$ for even n (figure 4 shows the computed $m_2'^2$ for different m_q taking $a = 0$). In particular, we cannot compute the mass of the lightest one, which is of physical importance since it would correspond to the $a_0(1450)$ meson [18]. A similar technical problem was discussed in [15], where it was overcome by an appropriate change of the choice of the coordinates describing the worldvolume and the excited scalar. In this case, we have not been able to find such a convenient change for general u_0 and the formalism gets too complicated. On the other hand, for the case $u_0 = u_\Lambda$ (with $a = 0$) it is indeed possible to find a convenient change of coordinates, in analogy with [18]. Details are reported in appendix A. The masses of the lightest states with $u_0 = u_\Lambda$ and $a = 0$ are:

$$
\begin{aligned}
m_1^2 &= 0.55 M_\Lambda^2 & m_2^2 &= 1.53 M_\Lambda^2 & m_3^2 &= 3.04 M_\Lambda^2 & m_4^2 &= 5.06 M_\Lambda^2 & \dots \\
m_1'^2 &= 2.28 M_\Lambda^2 & m_2'^2 &= 4.30 M_\Lambda^2 & m_3'^2 &= 6.82 M_\Lambda^2 & m_4'^2 &= 9.83 M_\Lambda^2 & \dots
\end{aligned}
\tag{3.30}
$$

Discussion. As all non-critical string models where the action has been truncated to terms with two derivatives, the model we are considering is expected to suffer from relevant curvature corrections. Nonetheless, symmetry arguments suggest that the underlying AdS structure of the space should provide some protection and make corrections smaller than expected, as we already noticed to be the case for the unflavored solution analyzed in [4]. For this reason it is still interesting to compare our predictions for the spectrum of mesons with the experimental values, and with the values obtained in the context of critical string theory in [18]. We summarize the results in table 1.

We have to notice, though, that curvature corrections to the present model are not the only kind of corrections we expect to arise. Just as the critical dimension models in [8, 18], the thermal modes, that is the KK modes surviving the anti-periodic boundary conditions projection over the S^1 circle η, have masses of the same order of magnitude as the lightest glueballs of the four-dimensional Yang-Mills theory. Therefore meson masses and scattering amplitudes will get contributions from loops containing these spurious modes, which do

	experiment	D4-D8 model	non-critical D4-D4
m_2^2/m_1^2	2.51	2.4	2.8
m_3^2/m_1^2	3.56	4.3	5.5
$m_1'^2/m_1^2$	3.61	4.9	4.1
$m_2^2/m_1'^2$	0.70	0.49	0.67

Table 1: Comparison of some experimental meson mass ratios with the results obtained from the model of [18] and the non-critical model with $a = m_q = 0$.

not belong to the spectrum of pure Yang-Mills. For the models in [8, 18], there are even more KK modes which do not belong to the spectrum of pure Yang-Mills: they come from the internal space over which ten-dimensional string theory is compactified. One of the advantages in using non-critical strings (even though partially overshadowed by the effects of stringy corrections) is that these modes are not present simply because there is no additional internal space over which the non-critical theory needs to be compactified.

3.4 Spinning open strings

Apart from the spectrum of the lightest mesons of the gauge theory, we can use the string/gauge duality to study the spectrum of mesons with very large spin. Let us consider the dynamics of an open string that has its endpoints attached to a probe flavor brane, and that rotates on a plane inside four-dimensional Minkowski. The interpretation of this string is that of a meson in the gauge theory, composed by a quark-antiquark pair. By analyzing the relationship between the energy and angular momentum of the spinning open string, we expect to be able to study the Regge trajectory of mesons in a QCD-like theory [8, 23, 29].

We leave most of the details of the calculation for appendix B, and collect here only the relevant results and comments. We consider, for clarity, a string centered at the origin of four-dimensional Minkowski space, spinning with constant angular velocity ω on the x^1, x^2 plane, which we parameterize as

$$ds^2 = dR^2 + R^2 d\phi^2 \tag{3.31}$$

The classical expressions for the energy and angular momentum of the open string are given in (B.8) and (B.9)

$$E = T_q \int d\sigma (1 + \rho^2)^{\frac{2}{5}} \sqrt{\frac{R'^2 + \frac{4}{25} \frac{R_{AdS}^4}{u_\Lambda^2} \frac{\rho'^2}{(1+\rho^2)^{7/5}}}{1 - \omega^2 R^2}} \tag{3.32}$$

$$J = T_q \omega \int d\sigma R^2 (1 + \rho^2)^{\frac{2}{5}} \sqrt{\frac{R'^2 + \frac{4}{25} \frac{R_{AdS}^4}{u_\Lambda^2} \frac{\rho'^2}{(1+\rho^2)^{7/5}}}{1 - \omega^2 R^2}} \tag{3.33}$$

where

$$T_q \equiv \frac{1}{2\pi\alpha'} \left(\frac{u_\Lambda}{R_{AdS}} \right)^2 = \frac{3}{5\pi} M_\Lambda^2 \tag{3.34}$$

is the effective tension for a string stretching close to the horizon of the AdS_6 black hole (2.1) [23]. In general, these expressions are largely affected by stringy corrections coming from the vibration of the string around its classical configuration. Nonetheless there is a particular limit where a semiclassical approximation is reliable, that is when the angular momentum J of the string is really large. In this case, the angular velocity of the string is really small and the distance between its two endpoints becomes very large. This configuration is really close to the static string representing the Wilson loop for two external quarks [30], and therefore the string can be effectively described as three straight segments: two vertical ones connecting the probe D4-brane to the horizon of the black hole, and an horizontal one lying on the horizon of the black hole. The string behaves like a large string rotating on the horizon of the black hole, with two massive quarks attached at its endpoints.

In this limit it is easy to show that the energy and angular momentum of the string are given by (B.22) and (B.23) respectively

$$E = \frac{2T_q}{\omega} \left(\frac{\sqrt{1 - \omega^2 R_0^2}}{\omega R_0} + \arcsin(\omega R_0) \right) \tag{3.35}$$

$$J = \frac{T_q}{\omega^2} \left(\omega R_0 \sqrt{1 - \omega^2 R_0^2} + \arcsin(\omega R_0) \right) \tag{3.36}$$

where the interquark distance $2R_0$ is determined in terms of the quarks dynamical mass and angular velocity ω through (B.21)

$$R_0 = \frac{m_q}{2T_q} \left(\sqrt{1 + \frac{4T_q^2}{m_q^2 \omega^2}} - 1 \right) \tag{3.37}$$

Therefore, in the limit of J very large where the semiclassical approximation is valid, the energy and angular momentum of the spinning open string satisfy a Regge law

$$E^2 = 2\pi T_q J = \frac{6}{5} M_\Lambda^2 J \tag{3.38}$$

Even though our analysis is not strictly reliable away from the large J region, it is still interesting to extrapolate the (energy)^2/angular momentum trajectory from (3.32) and (3.33) to lower values of J. The result is shown in figure 5. If the semiclassical analysis is at least approximately valid outside the large J region, the arguments above predict a deviation from the linear Regge behavior. A qualitative analysis of the small J limit is also reported at the end of appendix B.

Note that (3.38) means that the mass scale of the mesons on the Regge trajectories is the same as that of the massive mesons discussed in section 3.3, namely, M_Λ. This is not the case in the critical model of [18] where $T_q \sim M_\Lambda^2 g_{YM}^2 N_c$ which means that only for $g_{YM}^2 N_c \sim 1$ the mass scales of the mesons associated with the fluctuations of the probe branes and the spinning strings, can be the same. This obviousy implies curvature of order one, the same as in the non-critical model.
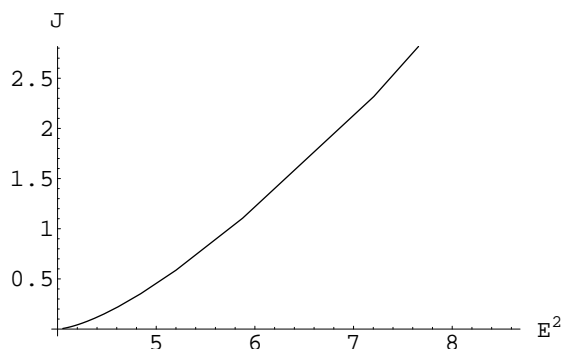
Figure 5: The plot shows an extrapolation to the small angular momentum region of formulae (3.32) and (3.33). Our approximate analysis shows that in this region the (E^2, J) trajectory deviates from the linear Regge behavior. Moreover, it is evident that there is a non-zero negative intercept.

3.5 On the mass of the quarks

We want to end this section with a brief discussion about the mass of the quarks as extracted from the analysis of the gravity background plus probe branes versus the notions of quarks masses from QCD and the quark model. Let us first remind ourselves the latter. In the QCD action bare masses for the various flavored quarks are introduced, which upon renormalization turn into physical masses, which sometimes are referred to as current algebra masses. For instance the u and the d quarks have masses of few MeV. Let us denote this mass as m_q^{QCD}. This mass parameter is very significant for many QCD processes. Here we focus only on one issue, that is the mass of the pion. It is well known that the mass square of the pion is proportional to m_q^{QCD} as follows

$$m_\pi^2 \sim (m_u^{QCD} + m_d^{QCD}) \tag{3.39}$$

where m_u^{QCD} and m_d^{QCD} are the physical masses of the u and the d quarks respectively. In addition, in the context of the quark model or potential models, one defines a constituent mass parameter which is roughly for the d and u quarks one third of the mass of the proton, namely around 300 MeV.

In the gravity plus probe brane model we consider, there are three pieces of information that can teach us which role the quark mass plays in this setup.

- The spectrum of massive mesons
  We found that on top of the Goldstone bosons there are massive vector and pseudoscalar mesons and that their masses increase monotonically[3] with the mass parameter m_q, defined in (3.5) as the energy of the string that stretches between the lowest point of the probe brane and the wall. In fact, from some mass scale of the order of M_Λ, the meson masses grow linearly in m_q (see figure 6), and the slope is bigger for higher states[4]. We therefore conclude that this mass parameter m_q is related to the constituent mass of the quark.

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[3]For $a \neq 0$ this is the case only from some value of m_q on.

[4]In the supersymmetric case of [8], the meson masses also grow linearly in the quark mass.
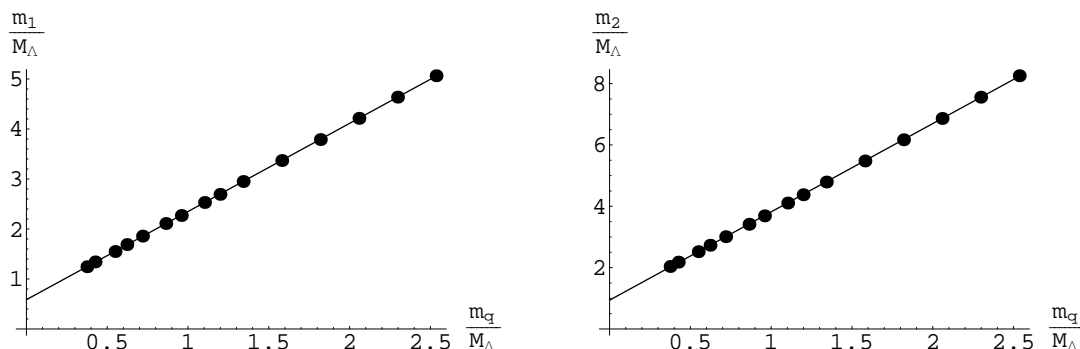
Figure 6: Asymptotically, the meson masses grow linearly in the parameter m_q. Here we present the plot for the two lightest vector mesons.

- The spectrum of large spin mesons
  As we have seen above, the spinning string that describes a meson of large spin, has a U shape, stretching from a probe brane vertically to the wall, along the wall and then again vertically to a probe brane (not necessarily the one it started from). The energy of each of the two vertical parts, which is exactly $m_q = E_s$, contributes to the total mass of the meson as a relativistic spinning particle, as can be seen when we rewrite the energy E of (3.35) in the following form

$$E = \frac{2m_q}{\sqrt{1 - w^2 R_0^2}} + \frac{2T_q}{\omega}\left(\arcsin(\omega R_0)\right) \tag{3.40}$$

This result agrees with our interpretation of m_q as related to the constituent mass of the quarks.

- The spectrum of Goldstone bosons
  As mentioned in section 3.1, the mass of the Goldstone bosons is always zero in our model, even when $u_0 \neq u_\Lambda$ and therefore m_q as defined in (3.5) is nonzero. On the contrary, we know that if the quarks have a bare mass, the chiral U(1) is not a symmetry of the lagrangian and the Goldstone boson should also acquire a mass, proportional to $\sqrt{m_q^{QCD}}$ for small m_q^{QCD}. This behavior was first found in a holographic setup in [14]. Thus we see that our gravity model seems to describe quarks with vanishing current algebra masses $m_u^{QCD} = m_d^{QCD} = 0$. To the best of our knowledge creating a current algebra mass in the models based on a brane anti-brane configuration is still an open question. Note that a non-vanishing constituent mass is possible even if the current algebra mass is zero. However since in QCD one cannot vary the constituent mass without changing the current algebra mass, there is an obvious puzzle in these results. In [18], following arguments of [31], it was argued that in the D4-D8-D̄8 critical setup a bare mass for the quarks should be related to an expectation value for the tachyon field coming from the open strings joining the flavor branes and anti-branes. Since the effective action we wrote does not account for these modes, it cannot describe this feature satisfactorily. It is reasonable to

conjecture that in the full stringy description of this system, there would not be a massless mode when $u_0 \neq u_\Lambda$, but unfortunately we lack a better understanding of this point.

4. Backreaction with large-N_f D4 flavor branes

It would be very interesting to be able to go beyond the probe approximation $N_f \ll N_c$. This would allow us to evaluate also those quantities of QCD which strongly depend on the dynamics of fundamental quarks. In this section we take a first step in this direction, by studying what happens when many flavor D4-branes are added to the AdS_6 black hole background of [2, 4].

The action we start from reads (we take $\alpha' = 1$)

$$S = \frac{1}{2k_6^2} \int d^6x \sqrt{-g} \left\{ e^{-2\phi} \left(R + 4\, \partial_\mu \phi\, \partial^\mu \phi + 4 \right) - \frac{1}{2 \cdot 6!}\, F_{(6)}^2 \right\} + S_{\text{flavors}} \qquad (4.1)$$

where k_6 is the 6-dimensional Newton's constant, and the action for the flavor D4-branes is given by:

$$S_{\text{flavors}} = T_4 \sum^{N_f} \left(- \int_{\mathcal{M}_5} d^5x\, e^{-\phi} \sqrt{-\hat{g}_5} + \tilde{a} \int_{\mathcal{M}_5} \mathcal{P}(C_{(5)}) \right) \qquad (4.2)$$

Here $\hat{g}_5$ is the induced metric over the D4-brane worldvolume $\mathcal{M}_5$, while $\mathcal{P}(C_{(5)})$ is the pull-back of the RR 5-form potential over the D4-branes worldvolume. $\tilde{a}$ is a constant which fixes the relative strength of the DBI and CS terms as in eq. (3.1). Unlike the probe case here due to the particular configuration we consider, with the flavor branes smeared over the η compact direction, it is clear that one has to set $\tilde{a}$ to zero. Notice also that taking a non-trivial value for $\tilde{a}$ would generate terms in the action which are not periodic under shifts $\delta\eta$ of η (cf. (2.5)), which is definitely not acceptable.

We look for backgrounds of the form

$$ds^2 = e^{2\lambda} dx_{1,3}^2 + dr^2 + e^{2\tilde{\lambda}} d\eta^2 \qquad (4.3)$$

and take the most symmetric assumption, that is all functions λ, $\tilde{\lambda}$ and the dilaton ϕ can only depend on the radius r. The color D4-branes wrap the circle η, while the flavor D4-branes are points on this circle. In order to simplify the following calculations, we take a U(1) symmetric distribution of the flavor D4-branes, that is we smear them uniformly over the circle, in a way analogous to [5]. The DBI term, reads then

$$-T_4 \sum^{N_f} \int d^5x\, e^{-\phi} \sqrt{-\hat{g}_5} = -T_4 \int d\eta \frac{N_f}{\delta\eta} \int d^5x\, e^{-\phi} \sqrt{-\hat{g}_5} = -\frac{T_4\, N_f}{\delta\eta} \int d^6x\, e^{-\phi-\tilde{\lambda}} \sqrt{-g_6} \qquad (4.4)$$

It is now trivial to write the equation of motion for the 6-form field strength

$$d * F_{(6)} = 0 \qquad (4.5)$$

which therefore requires

$$* F_{(6)} = Q_c \qquad (4.6)$$

– 18 –

where Q_c is a constant. It is straightforward to show that the following 5-form potential is a good solution to (4.6)

$$C_{(5)} = \beta(r) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\eta \tag{4.7}$$

with

$$\beta'(r) = Q_c \, e^{4\lambda + \tilde{\lambda}} \tag{4.8}$$

We are now ready to write the effective action for the ansatz (4.3). We skip a few intermediate steps (integration by parts and some algebra), and display the result

$$S_{\text{eff}} = \frac{4\pi \text{Vol}(\mathbb{R}^{1,3})}{k_6^2} \int dr \, e^{4\lambda + \tilde{\lambda} - 2\phi} \Big(3\lambda'^2 + 2\lambda'\tilde{\lambda}' - 4\lambda'\phi' - \tilde{\lambda}'\phi' + \phi'^2 + 1$$
$$- \frac{Q_f}{4} e^{\phi - \tilde{\lambda}} - \frac{Q_c^2}{8} e^{2\phi} \Big) \tag{4.9}$$

where we defined $Q_f = 2\frac{T_4 k_6^2}{\delta\eta} N_f$.

As in [2], we define a deformed dilaton

$$\varphi = -2\phi + 4\lambda + \tilde{\lambda} \tag{4.10}$$

and a new radial coordinate ρ such that $d\rho = -e^{-\varphi} dr$. The effective lagrangian reads, then, (a dot stands for derivation with respect to ρ)

$$\mathcal{L} = \dot{\varphi}^2 - 4\dot{\lambda}^2 - \dot{\tilde{\lambda}}^2 + 4e^{2\varphi} - \frac{Q_c^2}{2} e^{\varphi + 4\lambda + \tilde{\lambda}} - Q_f \, e^{\frac{3}{2}\varphi + 2\lambda - \frac{1}{2}\tilde{\lambda}} \tag{4.11}$$

The zero-energy condition reads

$$\dot{\varphi}^2 - 4\dot{\lambda}^2 - \dot{\tilde{\lambda}}^2 = 4e^{2\varphi} - \frac{Q_c^2}{2} e^{\varphi + 4\lambda + \tilde{\lambda}} - Q_f \, e^{\frac{3}{2}\varphi + 2\lambda - \frac{1}{2}\tilde{\lambda}} \tag{4.12}$$

while after going back to the original dilaton ϕ, the equations of motion read

$$\ddot{\phi} = -2e^{-4\phi + 8\lambda + 2\tilde{\lambda}} + \frac{3}{4} Q_c^2 e^{-2\phi + 8\lambda + 2\tilde{\lambda}} + \frac{3}{4} Q_f e^{-3\phi + 8\lambda + \tilde{\lambda}} \tag{4.13}$$

$$\ddot{\lambda} = \frac{Q_c^2}{4} e^{-2\phi + 8\lambda + 2\tilde{\lambda}} + \frac{Q_f}{4} e^{-3\phi + 8\lambda + \tilde{\lambda}} \tag{4.14}$$

$$\ddot{\tilde{\lambda}} = \frac{Q_c^2}{4} e^{-2\phi + 8\lambda + 2\tilde{\lambda}} - \frac{Q_f}{4} e^{-3\phi + 8\lambda + \tilde{\lambda}} \tag{4.15}$$

These equations are really involved and finding an explicit solution has proven a useless effort, up to now. But there is a special class of solutions which can be easily derived, those with a constant dilaton $\phi = \phi_0$. The equation for ϕ (4.13) turns then into an algebraic constraint

$$
\begin{aligned}
e^{2\phi_0} &= \frac{8}{3Q_c^2} & Q_f = 0 \\
e^{\tilde{\lambda}} &= \frac{3Q_f e^{-\phi_0}}{8e^{-2\phi_0} - 3Q_c^2} & Q_f \neq 0
\end{aligned} \tag{4.16}
$$

The case $Q_f = 0$ was considered in [2, 4], and leads to the AdS_6 black hole solution (2.1). On the other hand, if $Q_f \neq 0$, one finds an $AdS_5 \times S^1$ solution which can be obtained by performing T-duality along the S^1 to the solution found in [3]:

$$e^{\phi_0} = \frac{2}{\sqrt{3}Q_c} \ , \qquad R_{AdS}^2 = 6 \ , \qquad R_{S^1} = \frac{\sqrt{3}}{2}\frac{Q_f}{Q_c} \ . \qquad (4.17)$$

5. Flavored $AdS_{n+1} \times S^k$ backgrounds and black holes

The same line of reasoning as the one we followed in the preceding section can be generalized to a much wider class of non-critical string backgrounds. It has been found in [2], that the two-derivative truncation of the non-critical string theory action admits solutions of the form $AdS_{n+1} \times S^k$. They can be interpreted as the backgrounds generated by stacks of color D$(n-1)$-branes in $D = n+k+1$ dimensions. Here we consider the effect of including also space-filling D$(n+k)$ flavor branes. This analysis was made in [27] for the particular case $D = 6$. We will not worry about the stability of the solutions we find: we think it is convenient to keep a general formalism, even though some of the solutions considered could prove to be unstable due to open string modes.

We start from the two-derivative approximation to the $D = n + k + 1$-dimensional non-critical string action in the string frame (with $\alpha' = 1$)

$$
\begin{aligned}
S = & \frac{1}{2k_D^2} \int d^D x \sqrt{-g} \left(e^{-2\phi}(R + 4\partial_\mu\phi\partial^\mu\phi + 10 - D) - \frac{1}{2(n+1)!}F_{n+1}^2 \right) + \\
& - N_f T_{D-1} \int d^D x e^{-\phi}\sqrt{-g}
\end{aligned}
\qquad (5.1)
$$

where the $e^{-2\phi}(10 - D)$ term is the usual central charge appearing when working off criticality, and the last term is the contribution from the N_f uncharged flavor branes. We look for configurations that have the following structure:

$$ds^2 = e^{2\lambda}dx_{1,n-2}^2 + e^{2\tilde{\lambda}}d\eta^2 + dr^2 + e^{2\nu}d\Omega_k \qquad (5.2)$$

where the D$(n-1)$-branes extend along the $x^0, \ldots, x^{n-2}$ directions, are wrapped around the compact direction η, and are transverse to a k-sphere. We take all functions, including the dilaton ϕ, to depend only on r. The equation of motion for the $(n+1)$-form F_{n+1} reads $d * F_{n+1} = 0$, which is solved by

$$F_{n+1} = Q_c e^{(n-1)\lambda + \tilde{\lambda} - k\nu} dr \wedge dx^0 \wedge dx^1 \wedge \ldots \wedge dx^{n-2} \wedge d\eta \qquad (5.3)$$

After substituting our ansatz (5.2) and (5.3) into (5.1), we find the effective action

$$
\begin{aligned}
S \sim \int d\rho \Big(& \dot{\varphi}^2 - (n-1)\dot{\lambda}^2 - \dot{\tilde{\lambda}}^2 - k\dot{\nu}^2 + k(k-1)e^{2\varphi-2\nu} + (10-D)e^{2\varphi} + \\
& - \frac{Q_c^2}{2}e^{\varphi+(n-1)\lambda+\tilde{\lambda}-k\nu} - Q_f e^{\frac{3}{2}\varphi+\frac{n-1}{2}\lambda+\frac{1}{2}\tilde{\lambda}+\frac{k}{2}\nu} \Big)
\end{aligned}
\qquad (5.4)
$$

where we have defined $Q_f = 2k_D^2 T_{D-1} N_f$, introduced a generalized dilaton:

$$\varphi = -2\phi + (n-1)\lambda + \tilde{\lambda} + k\nu \qquad (5.5)$$

and used a new radial coordinate such that

$$d\rho = -e^{-\varphi} dr \qquad (5.6)$$

The equations of motion coming from this effective action read

$$\ddot{\phi} = -\frac{10-D}{2} e^{-4\phi + 2(n-1)\lambda + 2\tilde{\lambda} + 2k\nu} + \frac{n-k+1}{8} Q_c^2 e^{-2\phi + 2(n-1)\lambda + 2\tilde{\lambda}} +$$
$$+ \frac{D+2}{8} Q_f e^{-3\phi + 2(n-1)\lambda + 2\tilde{\lambda} + 2k\nu} \qquad (5.7)$$

$$\ddot{\lambda} = \frac{Q_c^2}{4} e^{-2\phi + 2(n-1)\lambda + 2\tilde{\lambda}} + \frac{Q_f}{4} e^{-3\phi + 2(n-1)\lambda + 2\tilde{\lambda} + 2k\nu} \qquad (5.8)$$

$$\ddot{\tilde{\lambda}} = \frac{Q_c^2}{4} e^{-2\phi + 2(n-1)\lambda + 2\tilde{\lambda}} + \frac{Q_f}{4} e^{-3\phi + 2(n-1)\lambda + 2\tilde{\lambda} + 2k\nu} \qquad (5.9)$$

$$\ddot{\nu} = (k-1) e^{-4\phi + 2(n-1)\lambda + 2\tilde{\lambda} + 2(k-1)\nu} - \frac{Q_c^2}{4} e^{-2\phi + 2(n-1)\lambda + 2\tilde{\lambda}} + \frac{Q_f}{4} e^{-3\phi + 2(n-1)\lambda + 2\tilde{\lambda} + 2k\nu} \qquad (5.10)$$

Finding the general solution to this coupled system of differential equations is far too difficult to hope to be able to get a sensible answer. Nonetheless, we can explicitly derive a class of special solutions, that is those with a constant dilaton $\phi = \phi_0$. This makes the first equation (5.7) into an algebraic equation relating the value of ϕ_0 with the function ν. In particular when $k \neq n+1$, (5.7) requires also ν to be constant $\nu = \nu_0$. For $k = n+1$ one could in general have a non-constant radius for the k-sphere. Nonetheless, we will stick to the simpler case $\nu = \nu_0$ also when $k = n+1$. Again, this makes the fourth equation (5.10) into another constraint on ϕ_0 and ν_0. We now denote the radius of the S^k and the string coupling as follows

$$R_S \equiv e^{\nu_0} \qquad g_s \equiv e^{\phi_0} \qquad (5.11)$$

After some algebra, one finds that R_S^2 satisfies an equation of degree $k+2$

$$(10-D) R_S^{2(k+2)} + (D+2)(k-1) R_S^{2(k+1)} +$$
$$- \frac{2}{n+2} \frac{Q_c^2}{Q_f^2} \left((10-D) R_S^2 - (n-k+1)(k-1) \right)^2 = 0 \qquad (5.12)$$

and the string coupling is given by

$$g_s = \frac{2}{Q_f} \frac{10 - D - (n-k+1)(k-1) R_S^{-2}}{n+2} \qquad (5.13)$$

Notice that when $Q_c \sim Q_f$, all the coefficients in equation (5.12) are order one, and therefore the radius of the k-sphere is a finite number while, from equation (5.13), the string coupling is order $1/Q_f$.

It is now straightforward to solve equations (5.8) and (5.9) for λ and $\tilde{\lambda}$

$$\lambda = -\frac{1}{n} \log\left(\frac{1}{b_1} \sinh\left(b_1 \sqrt{hn}\rho\right)\right) + a_1 \rho \tag{5.14}$$

$$\tilde{\lambda} = -\frac{1}{n} \log\left(\frac{1}{b_1} \sinh\left(b_1 \sqrt{hn}\rho\right)\right) - (n-1)a_1 \rho \tag{5.15}$$

where h is a parameter we have introduced to simplify formulas

$$h = \frac{Q_c^2}{4} e^{-2\phi_0} + \frac{Q_f}{4} e^{-3\phi_0 + 2k\nu_0} \ , \tag{5.16}$$

whereas a_1 and b_1 are two constants of integration. The other two have been fixed by taking a suitable origin for ρ and rescaling the coordinates $x^0, x^1, \ldots, x^{n-2}$ and η. A relation holds between the two remaining constants a_1 and b_1, and it comes from imposing the zero-energy condition

$$\dot{\varphi}^2 - (n-1)\dot{\lambda}^2 - \dot{\tilde{\lambda}}^2 - k\dot{\nu}^2 = \tag{5.17}$$
$$= k(k-1)e^{2\varphi - 2\nu} + (10-D)e^{2\varphi} - \frac{Q_c^2}{2} e^{\varphi + (n-1)\lambda + \tilde{\lambda} - k\nu} - Q_f e^{\frac{3}{2}\varphi + \frac{n-1}{2}\lambda + \frac{1}{2}\tilde{\lambda} + \frac{k}{2}\nu}$$

The result is the following relation

$$a_1 = b_1 \sqrt{\frac{h}{n}} \tag{5.18}$$

We found, therefore, a one-parameter family of solutions which are characterized by the dilaton and radius of the sphere transverse to the D$(n-1)$-branes being constant. As we will readily show these solutions are black hole deformations of $AdS_{n+1} \times S^k$ backgrounds which are obtained by taking $b_1 = 0$. Let us start from this simpler case. If we set $b_1 = 0$, the expressions for λ and $\tilde{\lambda}$ read:

$$\lambda = \tilde{\lambda} = -\frac{1}{n} \log\left(\sqrt{hn}\,\rho\right) = \sqrt{\frac{h}{n}} e^{2\phi_0 - k\nu_0} r \tag{5.19}$$

where an integration constant has been fixed in order to cancel a constant term. Thus for $b_1 = 0$ the solution is an $AdS_{n+1} \times S^k$ space, similar to those obtained in [2]. The radius of the AdS part is given by

$$\frac{n(n+2)}{R_{AdS}^2} = 10 - D + \frac{k^2 - 1}{R_S^2} \tag{5.20}$$

where R_S^2 is as given in (5.12).

Let us now consider several classes of special cases:

- The case that there is no sphere at all in the transverse directions, namely, where the metric is of the form $ds^2 = e^{2\lambda} dx_{1,n-2}^2 + e^{2\tilde{\lambda}} d\eta^2 + dr^2$ requires a special treatment. Equations (5.7)–(5.9) are still intact with $k = 0$, but obviously (5.10) does not exist anymore. The value of the string coupling is now

$$g_s = \frac{D+2}{2D} \frac{Q_f}{Q_c^2} \left[\sqrt{1 + \frac{16D(10-D)}{(D+2)^2} \left(\frac{Q_c}{Q_f}\right)^2} - 1\right] \tag{5.21}$$

and the radius of Anti de Sitter is given by

$$R_{AdS}^2 = \frac{4n}{g_s^2}[Q_c^2 + g_s^{-1}Q_f]^{-1} \tag{5.22}$$

- For the case of critical superstrings, $D = 10$ we get for $k > 1$

$$R_S^2 = \left(\frac{n+2}{24(k-1)}(g_sQ_c)^2\right)^{\frac{1}{k-1}} \tag{5.23}$$

and R_{AdS}^2 is related to R_S^2 in the following way

$$\frac{n(n+2)}{R_{AdS}^2} - \frac{k^2-1}{R_S^2} = 0 \tag{5.24}$$

and g_s is given by

$$g_s^{\frac{k+1}{k-1}}Q_f = 2\frac{(k-1)(k-n-1)}{n+2}\left(\frac{24}{Q_c^2}\frac{k-1}{n+2}\right)^{\frac{1}{k-1}} \tag{5.25}$$

In particular for the case with no flavors we get that the only solution is for $n = k - 1 = 4$ which is the $AdS_5 \times S^5$ solution. With flavors there is a solution only provided that $k > (n+1)$. This implies that there is no flavored $AdS_5 \times S^5$ solution associated with space filling branes. On the other hand there are solutions of the form $AdS_4 \times S^6$, $AdS_3 \times S^7$ and $AdS_2 \times S^8$ which may be interpreted as the flavored near horizon limit of D2, D1 and D0 branes respectively.

- A special class of non-critical solutions is for the case of an S^1, namely, $k = 1$ where

$$g_s = \frac{2(10-D)}{n+2}\frac{1}{Q_f} \qquad R_{AdS}^2 = \frac{D(D-2)}{10-D} \qquad R_S^2 = 2\frac{10-D}{n+2}\frac{Q_c^2}{Q_f^2} \tag{5.26}$$

In particular for $k = 1, n = 4, D = 6$ we get the KM solution [3].

- For the special class where $k = n + 1$ we get

$$g_s = \frac{2(10-D)}{n+2}\frac{1}{Q_f} = \frac{4(10-D)}{D+2}\frac{1}{Q_f} \tag{5.27}$$

The relation between the radii is

$$\frac{1}{R_{AdS}^2} - \frac{1}{R_S^2} = \frac{10-D}{n(n+2)} \tag{5.28}$$

Let us now proceed and describe the corresponding black hole solutions where $b_1 \neq 0$. We introduce a new radial coordinate u
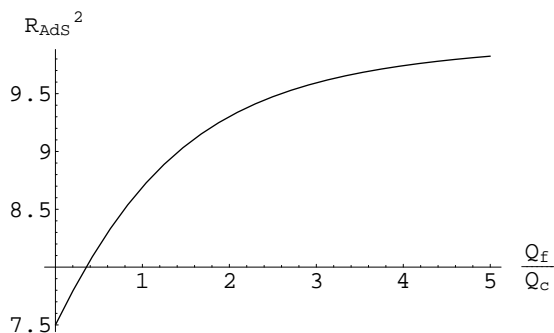
$$\frac{u}{R_{AdS}} = e^\lambda \tag{5.29}$$

Figure 7: Variation of R^2_{AdS} with the ratio of flavors to colors in the flavored AdS_6 black hole.

In terms of this new variable, the solution reads

$$ds^2 = \left(\frac{u}{R_{AdS}}\right)^2 \left(dx^2_{1,n-2} + \left(1 - \left(\frac{u_\Lambda}{u}\right)^n\right) d\eta^2\right) + \left(\frac{R_{AdS}}{u}\right)^2 \frac{du^2}{1 - \left(\frac{u_\Lambda}{u}\right)^n} + e^{2\nu_0} d\Omega_k \quad (5.30)$$

where u_Λ is the position of the horizon of the black hole

$$u_\Lambda^n = 2b_1 R_{AdS}^n \tag{5.31}$$

Notice that when $b_1 > 0$, the metric component $g_{\eta\eta}$ vanishes at $u = u_\Lambda$. Therefore the background is defined only for $u \geq u_\Lambda$ and to avoid a conical singularity at the horizon, we have to impose that the coordinate η is periodic

$$\eta \sim \eta + \frac{4\pi R_{AdS}^2}{n\, u_\Lambda} \tag{5.32}$$

Before closing let us mention three particular solutions that are relevant via holography to interesting gauge theories.

5.1 Near extremal flavored AdS_6

This solution corresponds to adding uncharged spacetime filling branes to the background of sections 2 and 3. As argued in section 2, adding such D5-branes corresponds to adding fundamental flavors to a gauge theory at a finite temperature.

One can read from (5.21) and (5.22) the values of the dilaton and R_{AdS}:

$$e^{\phi_0} = \frac{2}{3} \frac{Q_f}{Q_c^2} \left(\sqrt{1 + 6\frac{Q_c^2}{Q_f^2}} - 1\right), \qquad R_{AdS}^2 = \frac{90}{12 + \frac{Q_f^2}{Q_c^2} - \frac{Q_f^2}{Q_c^2}\sqrt{1 + 6\frac{Q_c^2}{Q_f^2}}} . \tag{5.33}$$

It is interesting that R_{AdS} depends on the ratio of number of colors and flavors (see figure 7). In the context of the AdS_5 background, it was conjectured in reference [5] that certain values of R_{AdS} may bound the stable gravity solutions, leading to a conformal window in the field theory bounded by certain N_f/N_c ratios.

5.2 Near extremal KM model

There is a particularly interesting case, among all those considered in the main part of this section: the $AdS_5 \times S^1$ black hole. The setup is the same as the one in [3]: we consider a system of D3 and uncharged D5-branes in six dimensional non-critical string theory. But here, we compactify one of the gauge theory flat directions on a thermal cycle, in order to break supersymmetry. The resulting background is dual to a four-dimensional field theory at a finite temperature with fundamental flavors, which becomes effectively three-dimensional for high values of the temperature. This is the adaptation of the idea of Witten [12] to a theory in three dimensions with the important novelties that here we consider the set-up in non-critical string theory, as it has been done in [2, 4], and moreover that our starting theory contains fundamental degrees of freedom.

The background reads

$$ds^2 = \left(\frac{u}{R_{AdS}}\right)^2 \left(dx_{1,2}^2 + \left(1 - \left(\frac{u_\Lambda}{u}\right)^4\right) d\eta^2\right) + \left(\frac{R_{AdS}}{u}\right)^2 \frac{du^2}{1 - \left(\frac{u_\Lambda}{u}\right)^4} + R_{S^1}^2 d\theta^2 \quad (5.34)$$

with $R_{AdS} = \sqrt{6}$, $R_{S^1}^2 = \frac{4Q_c^2}{3Q_f^2}$, $u_\Lambda^4 = 2b_1 R_{AdS}^4$ and $e^{\phi_0} = \frac{4}{3Q_f}$. The RR 5-form field strength reads

$$F_5 = Q_c \left(\frac{u}{R_{AdS}}\right)^3 dx^0 \wedge dx^1 \wedge dx^2 \wedge d\eta \wedge du \quad (5.35)$$

and its dual is $\partial\chi = Q_c \, d\theta$. The period of the compact direction η is given by

$$\eta \sim \eta + \frac{\pi R_{AdS}^2}{u_\Lambda} \quad (5.36)$$

5.3 Near extremal flavored AdS_5 model

The solutions above also include a non-extremal flavored AdS_5 model, generalizing the five-dimensional construction of [5]. The metric is the same as that in (5.34) apart from the fact that now the S^1 term is missing. The value of the string coupling is[5]

$$e^{\phi_0} = \frac{7}{10} \frac{Q_f}{Q_c^2} \left[\sqrt{1 + \left(\frac{20Q_c}{7Q_f}\right)^2} - 1 \right] \quad (5.37)$$

and R_{AdS}^2 is given by:

$$R_{AdS}^2 = \frac{\frac{400}{7}}{\frac{100}{7} + \frac{Q_f^2}{Q_c^2} \left[1 - \sqrt{1 + \left(\frac{20Q_c}{7Q_f}\right)^2} \right]} \quad (5.38)$$

6. Gauge dynamics from the flavored near extremal models

We can now study the phenomenology of four-dimensional gauge dynamics from three models that were derived in the previous sections:

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[5]Notice that the difference with the expressions in [5] comes from different definitions of Q_c and Q_f.

	$g_s Q_c$	R^2_{AdS}
unflavored AdS_6	$\sqrt{\frac{8}{3}}$	$\frac{15}{2}$
flavored AdS_6	$\frac{2}{3}\frac{Q_f}{Q_c}\left(\sqrt{1+6\frac{Q_c^2}{Q_f^2}}-1\right)$	$\dfrac{90}{12+\frac{Q_f^2}{Q_c^2}\left[1-\sqrt{1+6\frac{Q_c^2}{Q_f^2}}\right]}$
KM model	$\frac{2}{3}\frac{Q_c}{Q_f}$	6
unflavored AdS_5	2	4
flavored AdS_5	$\frac{7}{10}\frac{Q_f}{Q_c}\left[\sqrt{1+(\frac{20Q_c}{7Q_f})^2}-1\right]$	$\dfrac{\frac{400}{7}}{\frac{100}{7}+\frac{Q_f^2}{Q_c^2}\left[1-\sqrt{1+\left(\frac{20Q_c}{7Q_f}\right)^2}\right]}$

Table 2: Comparison of the 't Hooft parameter $g_s Q_c$ and the AdS radius between the various models.

1. The flavored near extremal AdS_6 model

2. The near extremal KM model

3. The flavored near extremal AdS_5 model.

The first two models are six-dimensional and the last one is five-dimensional. The construction of the first and the third models can be viewed as either following the path

$$AdS \quad \rightarrow \quad \text{near extremal } AdS \quad \rightarrow \quad \text{flavored near extremal } AdS$$

or

$$AdS \quad \rightarrow \quad \text{flavored } AdS \quad \rightarrow \quad \text{near extremal flavored } AdS$$

whereas the second model, since it does not have an unflavored predecessor, necessarily follows

$$\text{flavored } AdS_5 \times S^1 \quad \rightarrow \quad \text{near extremal flavored } AdS_5 \times S^1$$

Before extracting gauge dynamical properties, let us identify the properties of the backgrounds that affect them. The backgrounds are characterized by the AdS radius, the value of the constant dilaton, the radius of the thermal circle, the flux of the "colored" RR form $\sim Q_c$ and the number of flavor branes Q_f. Table 2 includes the values of these features of the models. For completeness we added also the values of the unflavored cases.

Note that as for any other non-critical model, here too the curvature is order one in units of α' and cannot be reduced by taking either Q_c or Q_f to be large. Thus, as expected, we are facing for these models the generic curvature problem. On the other hand we can always be in a regime where the string coupling g_s is small. Obviously for the first and third models this happens by taking Q_c to be large and for the second Q_f to be large. Note however that the claim is that this model [3] is valid only for $Q_c \sim Q_f$ and hence again taking large number of colors guarantees small string coupling.

At this point the first question we need to address is what phase of the gauge theory dynamics can be associated with each of these gravity models.

- The starting point of the first model can be taken to be the unflavored conformal AdS_6 background. The latter is supposed to be the dual of a fixed point of a non-supersymmetric five-dimensional gauge theory without fundamental quarks. The vanishing of the beta function may be due to the existence of adjoint matter or it might be the "fixed point" of the theory at the far UV where the theory is free. Since the latter has a set of infinitely many high spin conserved charges, it does not seem to correspond to the AdS_6 background and hence it can be only the former option. Note however that in five dimensions such fixed points are not known and may even be forbidden. When we add flavors the existence of the flavored AdS_6 background means that there is still an IR fixed point which presumably will move with respect to the unflavored one. Next we turn on near extremality and dualize in fact a four-dimensional low energy effective gauge theory by using the mechanism of [12] of taking the radius of the thermal cycle to be small. The resulting gauge theory has a massless sector which is identical to that of the pure four-dimensional Yang Mills theory regardless of what nature the original fixed point had. This sector will be realized in the IR theory in the form of glueballs. In addition there are quarks with a mass of the order of the temperature which will form mesons in the IR theory. Those of course are absent in the unflavored theory.

- The second model is the KM model which before turning on near extremality is conjectured to describe the IR fixed point of the $N = 1$ SQCD. Moreover, there is no unflavored $AdS_5 \times S^1$ solution and hence it is clear that unlike the first case here a vanishing beta function can be achieved only with flavors. If indeed this is the case, namely it is the holographic dual of the IR fixed point of the $N = 1$ SQCD, then the near extremal solution corresponds to turning on finite temperature in the four-dimensional theory which breaks both supersymmetry and conformal invariance. The limit of large temperature is dual in the IR to a three-dimensional gauge theory.

- The extremal case of the third model corresponds to a flavored four-dimensional non-supersymmetric gauge theory at a fixed point. Based on the arguments given for the first case, we conjecture that it corresponds to an IR fixed point of QCD in the flavored case. In four dimensions there is no problem with such a fixed point. The black hole solution in this case corresponds to finite temperature four-dimensional QCD. Again in the limit of large temperature we end up with a three-dimensional gauge theory with massless gauge fields and very massive quarks.

The next question that we would like to investigate is what the properties of the dual gauge theory which can be extracted from the string side of the gauge/string duality are. We will address here the issues of the Wilson line, the low lying glueball and meson spectra, which are extracted from the gravitational background, and the higher spin glueball and meson states which follow from closed and open spinning string configurations.

6.1 The Wilson line

The Wilson line, namely, a string that connects an external quark anti-quark pair in the background of a near extremal AdS solution, was investigated in [32] and [30]. The former paper discussed the case of a theory at finite temperature while in the latter the focus is on the low-energy gauge theory at one less dimension which, in particular, was shown to admit an area law behavior for the Wilson loop. One of the examples considered is three-dimensional pure Yang Mills associated with the $AdS_5 \times S^5$ black hole. Actually, the result applies to other near extremal backgrounds not necessarily of AdS type like the near extremal D4 branes background that admits a confining Wilson loop in four-dimensional YM theory. Let us start with this case and then we will address the finite temperature behavior. The same situation as in [30] happens with the flavored near extremal solutions that we discuss here in the first and second models. The string stretches from its endpoints down to the wall and then along the wall and up again. The corresponding energy of such a string is given by [4]

$$E = T_q L - 2\kappa + \mathcal{O}\left((\log L)^\gamma e^{-\hat{a}L}\right) \tag{6.1}$$

where T_q is as defined in (3.34)

$$T_q = \frac{g_{00}(0)}{2\pi\alpha'} = \frac{1}{2\pi\alpha'}\left(\frac{u_\Lambda}{R_{AdS}}\right)^2, \tag{6.2}$$

and for the AdS_6 case $\hat{a} = \sqrt{5}\frac{u_\Lambda}{R_{AdS}^2}$, γ is a positive constant and $\kappa \approx 0.309 \frac{u_\Lambda}{2\pi}$. Note also that $u_\Lambda \sim M_\Lambda$ see (2.6).

However, the situation here is very different from that of the unflavored case. To see this imagine that we start with a small separation between the external quark and anti-quark and then gradually increase it. Up to a distance L_{break} we have a worldsheet connecting the worldlines of the two external quarks. At that point it is energetically favorable for the system to create a dynamical quark anti-quark pair, namely a string of vanishing length, and break the worldsheet into two parts one connecting the original external quark and the dynamical anti-quark and the second one connecting the dynamical quark to the external anti-quark. Now increasing further the separation distance between the original pair will not change this picture. The breaking length is determined from

$$T_q L_{\text{break}} \sim M_\Lambda \quad \rightarrow \quad L_{\text{break}} \sim \frac{1}{M_\Lambda} \tag{6.3}$$

where we made use of the fact that the energy of the "dynamical meson" $\sim M_\Lambda$ and that $T_q \sim M_\Lambda^2$. This is obviously the stringy manifestation of the well known fact that the Wilson loop of a theory with fundamental quarks breaks up.

The Wilson loop of the third model for the case of small near extremality corresponds to that of four-dimensional QCD theory at finite temperature. In [32] it was found that for distances of order $L \ll \frac{1}{\mathcal{T}}$, where $\mathcal{T}$ is the temperature, the energy as a function of the distance and temperature is given by

$$E = \frac{R_{AdS}^2}{\Gamma(\frac{1}{4})}\frac{1}{L}[1 + c(\mathcal{T}L)^4] \tag{6.4}$$

where c is a positive constant independent of the parameters of the background, whereas for $L \gg \frac{1}{T}$ the quarks become free and the binding energy vanishes. The latter case is due to the screening caused by the thermal bath. The difference between our case and that of the near extremal $AdS_5 \times S^5$ model discussed in [32], is just the value of R_{AdS} but the basic behavior is the same.

6.2 The glueball spectrum

The glueball spectra is extracted [6] from the spectrum of the fluctuations of the dilaton, the metric and the RR one-form which are computed in a linearized limit of the gravity action. In the framework of the unflavored non-critical AdS_6 black hole, this calculation was performed in [4]. Since we are interested in the four-dimensional case, we discuss here only the flavored near extremal AdS_6 solution among the set of solutions we have found. In fact, what is left over to do is just to introduce the modifications to the results of [4] due to the presence of the flavor term in the action. The equations of motion of the fluctuating fields are given in (3.2)–(3.4) of [4]. The flavor term does not affect the equation of the RR form but does modify that of the metric and the source term of the Laplacian of the fluctuation of the dilaton. Let's discuss the latter. The shift of the dilaton equation of motion takes the following form

$$\nabla \delta \phi = 4 e^{\phi_0} \delta \phi \rightarrow \nabla \delta \phi = 4(e^{\phi_0} - Q_f e^{2\phi_0}) \delta \phi \tag{6.5}$$

where $\delta \phi$ is the dilaton fluctuation and e^{ϕ_0} on the left-hand and right-hand sides are those of the unflavored and flavored solutions respectively.

Notice that there is a critical value of Q_f, or more precisely $\frac{Q_f}{Q_c}$, where the source term vanishes. It is obvious from the equation above that the condition for that is that $1 - Q_f e^{\phi_0} = 0$ which by inserting the value of the dilaton in (5.33) yields $\frac{Q_f}{Q_c} = \frac{\sqrt{3}}{2}$. For Q_f larger than the critical value the sign of the source is flipped. The study of this case is left for future investigation. Here we assume that Q_f is below the critical value. The corresponding spectra of the $n^{++} = 0^{++}, 1^{++}$ and are given by

$$M^2_{\mathbf{n}^{++},\phi} \approx \frac{A}{\beta^2} k(k + B_n) + O(k^0) \quad \text{with} \quad \beta = \frac{4\pi R^2_{AdS}}{5u_\Lambda}. \tag{6.6}$$

where A is a numerical factor that depends on $\frac{Q_f}{Q_c}$ (for $Q_f = 0$ it is 39.66), and $B_0 = 5.02$, $B_1 = \frac{1}{2}$ as extracted from the dilaton, one-form and metric respectively. For the derivation of the spectrum and a comparison with spectra deduced from both critical near extremal D4 branes as well as lattice computations see [4].

6.3 Spinning closed strings

The spectrum of glueballs of spin greater than 2 cannot be determined from the gravity background but rather only from spinning closed string configurations. The analysis of classical spinning closed strings in confining backgrounds was done in the context of the critical backgrounds in [20, 33, 22] and for the non-critical unflavored AdS_6 in [4]. The situation here is very similar to the later. It is straightforward to show that the configuration

similar to that of the form (B.3) (but fixing $\rho = 0$) is a solution of the classical equations of motion. In fact it was shown in [4] that the conditions to have a spinning classical configuration are in one to one correspondence with the conditions for the background to have an area law Wilson loop. This classical spinning string admits a Regge behavior with

$$J = \frac{E^2}{4\pi T_q} \tag{6.7}$$

Next we can incorporate quantum fluctuations around the classical spinning string configuration associated with the Regge trajectory. This in general leads to the famous intercept. This incorporation was previously done for the critical confining backgrounds in [20] and for the non-critical AdS_6 black hole in [4]. The result found there takes the form

$$J = \frac{(E - z_0)^2}{4\pi T_q} - \frac{3}{24}\pi + \Delta_{\mathrm{f}}, \tag{6.8}$$

where z_0 is proportional to u_Λ and Δ_{f} is the contribution of the massless and massive fermionic modes which we do not know how to determine. From this expression we read the bosonic intercept $\alpha_0 = z_0^2/(4\pi T_q) - \frac{3}{24}\pi$. Note that in addition to the intercept there is also a term linear in E. A difference between the result in the critical model [22] and the non-critical one is in the proportionally factor between z_0 and u_Λ.

6.4 The meson spectrum

The spectrum of the pseudoscalar and vector mesons was analyzed in detail in section 3 using the probe D4 flavor branes. To derive the corresponding spectrum for the fully backreacted backgrounds of the flavored AdS black hole solutions and the near extremal KM model we can follow a very similar approach. Recall that the flavor term (4.2) is in fact the flavor brane worldvolume action. One should compute the system of linear equations coming from the coupled oscillations of the closed and open string degrees of freedom of an action of the kind (4.1). Nevertheless, up to quadratic order, the $F_{\mu\nu}$ oscillations do not mix with those of the background, so after integrating over the thermal circle we will find again an action which yields a discrete spectrum of massive vector mesons similar to the one found in section 3. However, having many flavor branes on top of each other, the $U(N_f)$ non-abelian action has to be used. Unlike what has been found for the D4 probe case, if flavor is introduced with D5s, we do not expect to have Goldstone bosons since as it was mentioned above the fundamental quarks acquire a large mass and decouple. Indeed, one can check explicitly that the mode analogous to the $\phi_{(0)}$ of section 3.1 (see eq. (3.13)) is not regular at the origin for the case of a flavor D5 and, therefore, it is not in the spectrum. In section 3 we have analyzed the dependence of the masses of the mesons as a function of the mass parameter $m_q = E_s$ that is associated with the location of the flavor brane. It is not clear how to introduce the analogous parameter for the fully backreacted backgrounds. Obviously we can also extract the spectrum of the pseudoscalar mesons from the fluctuations of the embedding in a similar manner to what was done with the flavored probe branes.

To draw the holographic picture of the mesons of spin higher than one, one needs to study spinning string configurations rather than the gravity modes. This type of analysis was done in section 3.4 for the case of D4 probe branes and resulted in a picture of spinning strings in flat spacetime with massive endpoints, which is very similar to what was derived earlier in [23]. In our case we have space-filling branes so that the endpoints of the open strings can reside at any point in spacetime. This is obviously different from the case of the probes. As already mentioned, it is also not clear how to introduce the mass parameter that was related to the constituent mass in the probe scenario.

7. Summary and open questions

In this paper we have addressed the issue of the non-critical holographic duality of gauge theories that incorporate quarks in the fundamental representation. This was done both by using flavor probe branes as well as by using certain fully backreacted gravity backgrounds. For the probe analysis we followed [18]. We put a set of N_f D4 anti-D4 branes in the background of the near extremal non-critical AdS_6 solution. We solved for the classical configuration of these probe branes and studied the fluctuations of both the gauge fields on the brane as well as of the embedding coordinates. The spectra of these fluctuations translate into the spectra of the vector and (pseudo) scalar mesons respectively. We compared the results to those measured in experiments as well as to the values found in [18] using a critical string model, and found that they are in reasonably good agreement with both. An important issue that we address is the Goldstone bosons associated with the flavor chiral symmetry breaking. We computed the dependence of the meson masses on the "mass parameter" which is the minimal distance between the probe brane and the "wall" of the background.

We then moved on to the second approach. We derived a class of flavored near extremal $AdS_{n+1} \times S^k$ solutions. In particular, we wrote down the flavored AdS_6 and AdS_5 black hole solutions and the near extremal version of the KM model. We addressed the question of what gauge theory phases the gravity models describe. We presented conjectures about this question which we backed up with the calculations of Wilson loops and the spectrum of glueballs. We also discussed the meson spectrum.

Our perspective has been dual throughout the whole paper. On one side, we focused on the holographic description of dynamical flavors. It is worth pointing out advantages and disadvantages of the probe approach presented in section 3 with respect to that of the so-called AdS/QCD duality [16]: in both cases a simplifying assumption is taken, which does not hold a priori. In our case it is the two-derivative approximation to non-critical string theory on a highly curved background. In [16], the flavored degrees of freedom are dualized by considering different fields in an AdS space imposed by hand (not coming from some set of Einstein equations) with an *ad hoc* IR cutoff in order to enforce confinement. Having AdS, this approach allows a better holographic description of, for instance, the quark masses, the quark condensate and the GMOR relation. The understanding of these features is still an open problem for our model and for the critical one of [18]. On the other hand, we believe that the qualitative advantage of our approach is the fact that we

are building our (unjustified) approximation on a justified and first-principle theory: non-critical string theory, and one may hope that progress in understanding this theory will naturally lead to an improvement of our approximation. In this sense, our setup is similar to [18]. With respect to that work, the main advantage of the non-critical construction is the disappearance of KK-modes, which is replaced by the problem of large curvature. Notice that trying to decouple KK-modes in critical string theory also leads naturally to highly curved space-times. It is appealing that all these different approaches lead to qualitative agreement with several QCD features. This aspect relates to the second perspective we took in the paper, that is to test how reliable the two-derivative approximation we employ is. We found, for example, that already in our approximation, we could identify the same mechanism as in [18] to provide a natural description of the spontaneous breaking of chiral symmetry for massless quarks. The different models also lead to a rough matching of some experimental quantities. We want to stress that these facts are pointing out that the idea of a higher-dimensional description of QCD has a solid basis.

This project is just one step in the long journey of discovering the string theory dual of QCD. There are still several conceptual problems with the present approach as well as certain open questions. Among these questions we find:

- As we have mentioned several times throughout the paper, the low-energy gravity (or supergravity) limit of non-critical string theory is problematic since it is characterized by order one curvature which means that higher order curvature corrections are a priori not negligible. In [2] certain arguments were made to back up the conjecture that models of the form $AdS_{n+1} \times S^k$ are robust against higher curvature corrections, even though the corresponding radii are corrected. Here since we are discussing near extremal solutions these arguments do not hold anymore. In fact it is known that already in the critical models the near extremal backgrounds get corrections. Nonetheless, we believe that the property of the black hole solutions we found, of having a wall with a non-trivial string tension along it, which guarantees a confinement behavior [28], will not change due to higher order curvature corrections. Proving this conjecture is left to future investigation.

- The identification of a mechanism that generates a current algebra mass to the quarks and hence renders the Goldstone bosons massive is far from being understood. In particular we showed that the Goldstone bosons remain massless even when a non-trivial quark mass parameter is introduced. We intend to continue the investigation of this question.

- In addition to the mass of the quarks there are several properties on the gauge theory side that have not been yet extracted from the gravity side with the flavor branes. In particular, the expectation value of $\bar{\psi}\psi$ was derived in the model based on D6 branes [11] together with its dependence on the quark mass. This has not been done yet for models based on branes anti-branes configurations, such as those we have been considering.

- In recent years there has been an intensive effort to reveal the relation between gauge theories at finite temperature and their holographic gravity duals, with particular interest in the confining-deconfining phase transition etc. So far everything was done in the context of critical gravity backgrounds. The derivation of a similar description for the non-critical scenarios is an interesting question that we intend to address in the future.

- On top of the spectrum, one would also like to compute the decay rates of the various types of mesons. First steps toward this goal were taken in [19] where decay rates of low lying mesons were computed. In [34] an analysis of the decay processes of mesons is addressed using the picture of spinning open strings. These two methods of computing decay rates can be also applied to the non-critical models discussed in this paper.

- As we have mentioned above one can extract the mesonic spectrum from the fully backreacted solution in a similar manner to what we did for the probe setups. We intend to come back to this issue and compute the corresponding meson spectra.

Acknowledgments

A. Scalar mesons with $u_0 = u_\Lambda$

In this appendix we find the equation that allows to compute the spectrum of scalar mesons in the non-critical D4-D4-D̄4 system in the case when the flavor D4-D̄4 probes are extended along u at a fixed value of η, *i.e* in the case when one takes $u_0 = u_\Lambda$ and $a = 0$. The discussion closely follows that in [18].

Let us define new coordinates for the (u, η) plane of the metric (2.1):

$$u^5 = u_\Lambda^5 + u_\Lambda^3 r^2 \ , \qquad \tilde{\eta} = \frac{2\pi}{\delta\eta}\eta = \frac{5u_\Lambda}{2R_{AdS}^2}\eta \ , \qquad (A.1)$$

so that the period of $\tilde{\eta}$ is 2π. The metric in the (u, η) plane reads, in the new coordinates:

$$\left(\frac{R_{AdS}}{u}\right)^2 \frac{du^2}{f(u)} + \left(\frac{u}{R_{AdS}}\right)^2 f(u)d\eta^2 = \frac{4}{25}\frac{R_{AdS}^2 u_\Lambda}{u^3}\left(\frac{u_\Lambda^2}{u^2}dr^2 + r^2 d\tilde{\eta}^2\right) =$$

$$= \frac{4}{25} \frac{R_{AdS}^2 u_\Lambda}{u^3} \left[(1 - h\, z^2)dz^2 + (1 - h\, y^2)dy^2 - 2h\, z\, y\, dz\, dy \right] \qquad (A.2)$$

where we have defined:

$$y = r \cos \tilde{\eta}\,, \qquad\qquad z = r \sin \tilde{\eta} \qquad\qquad (A.3)$$

and

$$h = \frac{1}{r^2} \left(1 - \frac{u_\Lambda^2}{u^2} \right) \qquad\qquad (A.4)$$

The probe brane is located at a fixed value of the angle $\tilde{\eta}$, say, $\tilde{\eta} = \frac{\pi}{2}$ so, in the new coordinates, the brane is extended in z at $y = 0$. We want to study small fluctuations around this embedding so we now consider $y(x^\mu, z)$ and just keep terms in y up to quadratic order. It is easy to verify that the action (3.1) is:

$$S_{D4} = -\tilde{T} \int d^4 x dz \left[u^{\frac{3}{2}} + \frac{2}{25} \frac{R_{AdS}^4 u_\Lambda}{u^{\frac{7}{2}}} \eta^{\mu\nu} \partial_\mu y\, \partial_\nu y + \frac{1}{2} \frac{u^{\frac{7}{2}}}{u_\Lambda^2} \left(\dot{y}^2 + h(y^2 - 2zy\dot{y}) \right) \right] \quad (A.5)$$

where a dot denotes derivative with respect to z and $\tilde{T} = T_4 e^{-\phi} \frac{2}{5} \frac{u_\Lambda^{3/2}}{R_{AdS}^3}$. Finally, in order to simplify this expression, let us define:

$$u_z(z) = (u_\Lambda^5 + u_\Lambda^3 z^2)^{\frac{1}{5}} \qquad\qquad (A.6)$$

and subtract a total derivative $\partial_z \left(u_z^{\frac{7}{2}} h\, z\, y^2 / (2u_\Lambda^2) \right)$ so we find:

$$S_{D4} = -\tilde{T} \int d^4 x\, dz \left[\frac{2}{25} \frac{R_{AdS}^4 u_\Lambda}{u_z^{7/2}} \eta^{\mu\nu} \partial_\mu y\, \partial_\nu y + \frac{7}{10} \frac{u_\Lambda}{u_z^{3/2}} y^2 + \frac{1}{2} \frac{u_z^{7/2}}{u_\Lambda^2} \dot{y}^2 \right]\,. \qquad (A.7)$$

We want to obtain the spectrum of normalizable small fluctuations associated to the action (A.7). From the four-dimensional point of view, they correspond to (pseudo) scalar excitations. It will be useful to define several quantities:

$$Z \equiv \frac{z}{u_\Lambda}\,, \qquad\qquad K \equiv \left(\frac{u_z}{u_\Lambda} \right)^5 = 1 + Z^2\,, \qquad\qquad (A.8)$$

In terms of these quantities, the action for the fluctuation reads:

$$S_{D4} = -\frac{4}{25} \tilde{T} \frac{R_{AdS}^4}{u_\Lambda^{\frac{3}{2}}} \int d^4 x dZ \left[\frac{1}{2} K^{-\frac{7}{10}} \eta^{\mu\nu} \partial_\mu y\, \partial_\nu y + \frac{M_\Lambda^2}{2} \left(\frac{7}{5} K^{-\frac{3}{10}} y^2 + K^{\frac{7}{10}} (\partial_Z y)^2 \right) \right]$$
$$(A.9)$$

Now, by expanding:

$$y(x^\mu, Z) = \sum_{n=1}^\infty \mathcal{U}^{(n)}(x^\mu) \rho_{(n)}(Z) \qquad\qquad (A.10)$$

the equation for the set of $\rho_{(n)}$ reads:

$$-K^{\frac{7}{10}} \partial_Z (K^{\frac{7}{10}} \partial_Z \rho_{(n)}) + \frac{7}{5} K^{\frac{4}{10}} \rho_{(n)} = \left(\frac{m_n'}{M_\Lambda} \right)^2 \rho_{(n)}\,, \qquad\qquad (A.11)$$

and the orthonormality condition is:

$$\frac{4}{25}\tilde{T}\frac{R_{AdS}^4}{u_\Lambda^{\frac{3}{2}}}\int dZ K^{-\frac{7}{10}}\rho_{(n)}\rho_{(m)} = \delta_{mn} \tag{A.12}$$

Notice that from equation (A.11), it is easy to find out the behavior of the $\rho_{(n)}$ at infinity:

$$\lim_{Z\to\infty}\rho_{(n)} \sim C_1 Z + \frac{C_2}{Z^{\frac{7}{5}}} \tag{A.13}$$

From (A.12) we see that normalizability of the mode requires $C_1 = 0$ thus yielding a discrete spectrum. Since (A.11) is invariant under $Z \to -Z$ the solutions have well defined (odd or even) parity under this transformation, leading to mesons of different parity and charge conjugation as discussed in [18]. There are, therefore, two possible boundary conditions at $Z = 0$:

$$\begin{aligned}
\partial_Z\rho_{(n)}(0) = 0 \quad &\to 0^{++} \quad &(\text{odd } n) \\
\rho_{(n)}(0) = 0 \quad &\to 0^{--} \quad &(\text{even } n)
\end{aligned} \tag{A.14}$$

The values obtained numerically by inserting these conditions on (A.11) and requiring normalizability are reported on the main text (3.30).

B. Spinning open string calculation details

In this appendix we report the calculations of the spectrum of large angular momentum mesons, evaluated form the dual non-critical background. The calculation is very similar to what has been done in some critical examples [8, 23].

We start from a set of coordinates for the transverse space to the D4 color branes, which is closely related to the one introduced in (A.1). Let us take

$$u^5 = u_\Lambda^5(1+\rho^2) \qquad \tilde{\eta} = \frac{5u_\Lambda}{2R^2}\eta \tag{B.1}$$

with respect to which the metric reads

$$ds^2 = \left(\frac{u_\Lambda}{R_{AdS}}\right)^2 (1+\rho^2)^{\frac{2}{5}}(-(dx^0)^2+(dx^3)^2+dR^2+R^2\,d\phi^2)+\frac{4}{25}R_{AdS}^2\left(\frac{d\rho^2}{1+\rho^2}+\frac{\rho^2\,d\tilde{\eta}^2}{(1+\rho^2)^{\frac{3}{5}}}\right) \tag{B.2}$$

Since we consider an open string rotating on the (x^1, x^2) plane, that is along the angle ϕ, we take the following ansatz for the motion of the string

$$\begin{aligned}
X^0 = \tau \qquad &\phi = \omega\tau \qquad &X^3 = const. \tag{B.3} \\
R(\sigma,\tau) = R(\sigma) \qquad &\rho(\sigma,\tau) = \rho(\sigma) \qquad &\tilde{\eta}(\sigma,\tau) = \tilde{\eta}(\sigma) \tag{B.4}
\end{aligned}$$

Moreover, the endpoints of the string need to be attached to the probe D4, that is

$$\rho(\pm \tfrac{\pi}{2}) = \rho(u_{D4}(\eta)) \tag{B.5}$$

where $u_{D4}(\eta)$ is the profile of the probe D4-brane (3.4). The Nambu-Goto action for this configuration reads then

$$S_{NG} = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-h_{\alpha\beta}} =$$

$$= T_q \int_{\Sigma} d^2\sigma (1+\rho^2)^{\frac{2}{5}} \sqrt{((\dot{X}^0)^2) - R^2\dot{\phi}^2) \left(R'^2 + \frac{4}{25}\frac{R_{AdS}^4}{u_\Lambda^2}\left(\frac{\rho'^2}{(1+\rho^2)^{\frac{7}{5}}} + \frac{\rho^2}{1+\rho^2}\tilde{\eta}'^2\right)\right)} \tag{B.6}$$

where $h_{\alpha\beta}$ is the string worldsheet metric induced by the background (B.2), and

$$T_q \equiv \frac{1}{2\pi\alpha'}\left(\frac{u_\Lambda}{R_{AdS}}\right)^2 \tag{B.7}$$

is the effective tension for a string stretching close to the horizon of the AdS_6 black hole (B.2) [23].

The action (B.6) is independent of X^0 and ϕ, therefore the following quantities are conserved

$$E = T_q \int d\sigma (1+\rho^2)^{\frac{2}{5}} \sqrt{\frac{R'^2 + \frac{4}{25}\frac{R_{AdS}^4}{u_\Lambda^2}\left(\frac{\rho'^2}{(1+\rho^2)^{7/5}} + \frac{\rho^2}{1+\rho^2}\tilde{\eta}'^2\right)}{1-\omega^2 R^2}} \tag{B.8}$$

$$J = T_q \omega \int d\sigma R^2 (1+\rho^2)^{\frac{2}{5}} \sqrt{\frac{R'^2 + \frac{4}{25}\frac{R_{AdS}^4}{u_\Lambda^2}\left(\frac{\rho'^2}{(1+\rho^2)^{7/5}} + \frac{\rho^2}{1+\rho^2}\tilde{\eta}'^2\right)}{1-\omega^2 R^2}} \tag{B.9}$$

The variation of the action with respect to $\tilde{\eta}'$ is given by

$$\delta S_{NG}|_{\delta\tilde{\eta}'} = \frac{4R_{AdS}^2}{50\pi\alpha'} \int_{\Sigma} d^2\sigma \frac{\rho^2}{(1+\rho^2)^{\frac{3}{5}}} \sqrt{\frac{1-\omega^2 R^2}{R'^2 + \frac{4}{25}\frac{R_{AdS}^4}{u_\Lambda^2}\left(\frac{\rho'^2}{(1+\rho^2)^{7/5}} + \frac{\rho^2}{1+\rho^2}\tilde{\eta}'^2\right)}} \tilde{\eta}'\delta\tilde{\eta}' \tag{B.10}$$

and since the action does not depend on $\tilde{\eta}$, for the above variation to vanish it must be either $\tilde{\eta}' = 0$ or $\rho = 0$. For $u_0 > u_\Lambda$ only the first condition can be satisfied, and moreover setting $\tilde{\eta}' = 0$ always lowers the energy, as can be easily seen from (B.8). Therefore from now on we take $\tilde{\eta}$ to be constant for the embedding of the open string.

The total variation of the open string action (B.6) reads, after integrating by parts,

$$
\begin{aligned}
\delta S_{NG} =& T_q \int_\Sigma d^2\sigma \left\{ -\left[\sqrt{\frac{R'^2 + \frac{4}{25}\frac{R^4_{AdS}}{u^2_\Lambda}\frac{\rho'^2}{(1+\rho^2)^{7/5}}}{1-\omega^2 R^2}}(1+\rho^2)^{\frac{2}{5}}\omega^2 R + \right.\right.\\
&\left.\left. + \partial_\sigma\left(\sqrt{\frac{1-\omega^2 R^2}{R'^2 + \frac{4}{25}\frac{R^4_{AdS}}{u^2_\Lambda}\frac{\rho'^2}{(1+\rho^2)^{7/5}}}}(1+\rho^2)^{\frac{2}{5}}R' \right) \right]\delta R + \right.\\
&\left. + \frac{4}{5}\left[\sqrt{\frac{1-\omega^2 R^2}{R'^2 + \frac{4}{25}\frac{R^4_{AdS}}{u^2_\Lambda}\frac{\rho'^2}{(1+\rho^2)^{7/5}}}}\frac{\rho}{(1+\rho^2)^{3/5}}\left(R'^2 - \frac{3}{25}\frac{R^4_{AdS}}{u^2_\Lambda}\frac{\rho'^2}{(1+\rho^2)^{7/5}} \right) + \right.\right.\\
&\left.\left. - \partial_\sigma\left(\sqrt{\frac{1-\omega^2 R^2}{R'^2 + \frac{4}{25}\frac{R^4_{AdS}}{u^2_\Lambda}\frac{\rho'^2}{(1+\rho^2)^{7/5}}}}\frac{R^4_{AdS}}{5u^2_\Lambda}\frac{\rho'}{1+\rho^2} \right) \right]\delta\rho \right\} +\\
&+ T_q \int d\tau \sqrt{\frac{1-\omega^2 R^2}{R'^2 + \frac{4}{25}\frac{R^4_{AdS}}{u^2_\Lambda}\frac{\rho'^2}{(1+\rho^2)^{7/5}}}}(1+\rho^2)^{\frac{2}{5}}\left(R'\delta R + \frac{4}{25}\frac{R^4_{AdS}}{u^2_\Lambda}\frac{\rho'\delta\rho}{(1+\rho^2)^{7/5}} \right)\bigg|^{\sigma=\frac{\pi}{2}}_{\sigma=-\frac{\pi}{2}}
\end{aligned}
$$

(B.11)

The boundary term in the last line fixes the boundary conditions for the endpoints of the open string. There are two ways to set this term to zero

$$
\omega R\left(\pm\frac{\pi}{2}\right) = 1 \qquad \text{or} \qquad \begin{cases} R'\,\delta R|_{\sigma=\pm\frac{\pi}{2}} = 0 \\ \rho'\,\delta\rho|_{\sigma=\pm\frac{\pi}{2}} = 0 \end{cases}
$$

(B.12)

It has been shown in [8] that for a configuration similar to ours, imposing that the string endpoints move at the velocity of light does not allow for both of them to be attached to the same brane. Therefore the first condition in (B.12) has to be discarded. We are only left, then, with the second set of boundary conditions. For the first condition in this set, we notice that the probe brane lies along the R direction, and therefore it is natural to impose $R' = 0$ while leaving δR arbitrary: the string attaches perpendicularly to the brane, but its endpoints can move freely over it. For the last condition, it is sufficient to notice that the minimal energy condition requires the endpoints of the string to be as close as possible to the horizon of the black hole, that is the string needs to be attached to the lowest point of the probe brane

$$
\rho\left(\pm\frac{\pi}{2}\right) = \rho(u_0) \equiv \rho_0
$$

(B.13)

and therefore

$$
\delta\rho\left(\pm\frac{\pi}{2}\right) = 0
$$

(B.14)

which satisfies the remaining boundary condition (B.12).

There are two possible convenient choices to gauge the σ parameterization invariance of the string worldsheet, either $R = \sigma$ or $\rho = \sigma$. In the first case, the equation of motion

coming from setting (B.11) to zero reads

$$\frac{\rho''}{\rho'} - \frac{11}{5}\frac{\rho'\rho}{1+\rho^2} - 5\frac{u_\Lambda^2}{R_{AdS}^4}(1+\rho^2)^{\frac{2}{5}}\frac{\rho}{\rho'} - \frac{\omega^2 R}{1-\omega^2 R^2}\left(1 + \frac{4}{25}\frac{R_{AdS}^4}{u_\Lambda^2}\frac{\rho'^2}{(1+\rho^2)^{7/5}}\right) = 0 \quad \text{(B.15)}$$

while fixing $\rho = \sigma$ one obtains

$$\frac{R''}{R'} + \frac{11}{5}\frac{\rho}{1+\rho^2} + 5\frac{u_\Lambda^2}{R_{AdS}^4}\rho(1+\rho^2)^{\frac{2}{5}}R'^2 + \frac{\omega^2 R}{R'(1-\omega^2 R^2)}\left(R'^2 + \frac{4}{25}\frac{R_{AdS}^4}{u_\Lambda^2}\frac{1}{(1+\rho^2)^{7/5}}\right) = 0 \tag{B.16}$$

Rotating long string analysis. These equations are quite involved and most probably finding an exact solution might prove impossible. Nonetheless a numerical solution can certainly be found, by imposing appropriate boundary conditions. For the moment let us focus, though, on an interesting limit. When the distance $2R_0$ between the open string endpoints is very large, the angular velocity needs to be really small (for fixed quark masses) and the string configuration resembles very much the one encountered in the Wilson loop calculation. To a fairly good extent, therefore, it can be approximated by a piecewise-straight string with two sides stretching along the ρ direction from the probe brane to the black hole horizon at $\rho = 0$ and a third piece lying along the R direction at $\rho = 0$. This configuration has to be supplemented by an additional set of boundary conditions, requiring that the two internal vertices of the string (we can always choose a parameterization in which they are at $\sigma = \pm\alpha$) connecting the "vertical" and "horizontal" straight pieces, have to lie on the horizon of the black hole

$$\rho(\pm\alpha) = 0 \quad \Rightarrow \quad \delta\rho|_{\sigma=\pm\alpha} = 0 \tag{B.17}$$

By substituting this simplifying ansatz in the Nambu-Goto action variation (B.11), the only surviving terms are

$$\delta S_{NG} = T_q \left\{ -\frac{2}{5}\frac{R_{AdS}^2}{u_\Lambda}\frac{\omega^2 R_0}{\sqrt{1-\omega^2 R_0^2}} \int d\tau \left(\int_{-\frac{\pi}{2}}^{-\alpha} d\sigma + \int_{\alpha}^{\frac{\pi}{2}} d\sigma \right) \left(\frac{\rho'}{(1+\rho^2)^{3/10}} \right) \delta R + \right.$$
$$\left. + 2\sqrt{1-\omega^2 R_0^2} \int d\tau \delta R \right\} \tag{B.18}$$

and stationarity of the action can be achieved only by imposing $\delta R(\sigma, \tau) = \delta R(\tau)$ and enforcing the relation

$$T_q(1 - \omega^2 R_0^2) = \omega^2 R_0 m_q \tag{B.19}$$

where, in these coordinates, the definition (3.5) reads:

$$m_q \equiv \frac{1}{2\pi\alpha'} \int_0^{\rho_0} d\rho \sqrt{-g_{00}g_{\rho\rho}} = \frac{1}{2\pi\alpha'}\frac{2u_\Lambda}{5} \int_0^{\rho_0} \frac{d\rho}{(1+\rho^2)^{3/10}} \tag{B.20}$$

Apart from the dependence of the effective string tension T_q on the ratio u_Λ/R_{AdS}, equation (B.19) is the same as the relation obtained in [23]. Equation (B.19) can be easily solved for R_0

$$R_0 = \frac{m_q}{2T_q}\left(\sqrt{1 + \frac{4T_q^2}{m_q^2\omega^2}} - 1\right) \tag{B.21}$$

We can now evaluate the energy and angular momentum of the string configuration in the limit of large R_0. We find

$$E = \frac{2T_q}{\omega} \left(\frac{\sqrt{1 - \omega^2 R_0^2}}{\omega R_0} + \arcsin(\omega R_0) \right) \tag{B.22}$$

$$J = \frac{T_q}{\omega^2} \left(\omega R_0 \sqrt{1 - \omega^2 R_0^2} + \arcsin(\omega R_0) \right) \tag{B.23}$$

where the interquark distance $2R_0$ is determined in terms of the quarks' mass and angular velocity ω through (B.21). These relations are solidly reliable only in the large J limit which, for fixed m_q, is the one corresponding to the small ω region. In this regime it is easy to show that $R_0 \simeq 1/\omega$ and the energy and angular momentum of the string configuration satisfy a Regge law

$$E^2 = 2\pi T_q J \tag{B.24}$$

Rotating short strings analysis. We analyze here the limit in which $\omega \to \infty$ and therefore $R_0 \to 0$. Since the string is very short, it does not really get affected by the particular background where it is embedded and we will recover a result very similar to that found in [8].

Let us consider:

$$\rho = \rho_0 + \delta\rho(R) \tag{B.25}$$

Keeping only leading order terms[6] in $\delta\rho$ we find from (B.15):

$$\delta\rho'' - K - \frac{\omega^2 R}{1 - \omega^2 R^2} \delta\rho' = 0 \tag{B.26}$$

where the constant K has been defined as:

$$K = 5 \frac{u_\Lambda^2}{R_{AdS}^4} (1 + \rho_0)^{2/5} \rho_0 \tag{B.27}$$

This equation of motion has to be supplemented with the initial condition $\delta\rho'(0) = 0$, coming from the $R \to -R$ symmetry of the problem. It is immediate to find:

$$\delta\rho' = \frac{KR}{2} + \frac{\arcsin(R\omega)}{2\omega\sqrt{1 - R^2\omega^2}} \tag{B.28}$$

Since we need $\delta\rho'(R_0) = \infty$, we have $R_0 \to \omega^{-1}$. Substituting in (B.8) and (B.9) one finds that at leading $\omega \to \infty$ order, $E \propto \omega^{-1}$ and $J \propto \omega^{-2}$ so there is again a Regge behavior

$$E^2 = 2\pi\tau_{\text{eff}} J \tag{B.29}$$

with:

$$\tau_{\text{eff}} = \frac{1}{2\pi\alpha'} \left(\frac{u_\Lambda}{R_{AdS}} \right)^2 (1 + \rho_0^2)^{2/5} \tag{B.30}$$

Notice that $\tau_{\text{eff}} > T_q$.

---

[6]We are assuming $1 \gg \frac{R_{AdS}^4}{u_\Lambda^2} \frac{\rho'^2}{(1+\rho^2)^7/5}$. This is true except near the endpoints where of course $\rho' \to \infty$. What happens here is analogous to [8].

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